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An algebraic fractal approach to Collatz conjecture

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– *Doncs el TFG el faig sobre la Conjectura de Collatz.*
– *Ah, però no era alcaldessa?*

CONVERSA REAL

Abstract

The Collatz conjecture is one of the most easy-to-state unsolved problems in Mathematics today. It states that after a finite number of iterations of the Collatz function, defined by $C(n) = \frac{n}{2}$ if n is even, and by $C(n) = 3n + 1$ if n is odd, one always gets to 1 with independence of the initial positive integer value. In this work, we give an equivalent formulation of the weak Collatz conjecture (which states that all cycles in any sequence obtained by iterating this function are trivial), based on another equivalent formulation made by Böhm and Sontacchi in 1978. To such purpose, we introduce the notion of algebraic fractals, integer fractals and boolean fractals, with a special emphasis in their self-similar particular case, which may allow us to rethink some fractals as a relation of generating functions.

Keywords

Collatz Conjecture, Generating Functions, Fractals, Algebraic Fractals

Contents

1. Introduction

1.1 Motivation

The Collatz conjecture is one of the most intriguing open problems in Mathematics. Named after German mathematician Lothar Collatz, its formulation is so simple that any 10 year old kid can understand it: take a number (integer and positive), and divide it by two if it is even, or multiply it by three and add one to the result if it is odd. The Collatz conjecture states that, for any starting value, by repeating this process for long enough, one will arrive to 1 at some point. However, behind such a simple-to-state problem, there is hiding a 50-year-old[1] mystery that is, at least apparently, extremely complex. Even Paul Erdős, one of the most prolific mathematicians in history, stated that "Mathematics is not ready for such problems".

The Collatz conjecture has also many other names, such as *Collatz problem*, *3x + 1 problem*, *3n + 1 problem*, *3n + 1 conjecture*, or *Ulam conjecture*. Although many papers have been written about the subject, there is not any promising idea, at least by now, that can lead us to think it is going to be solved any time soon. In fact, some mathematicians think that the conjecture is unsolvable, and actually, John H. Conway [11] proved in 1972 that a certain generalization of the problem is undecidable.

The Collatz problem seems to be isolated from the rest of Mathematics and can even be deemed as irrelevant, in the sense that whether it is true or false, the answer cannot be applied in other problems or branches of Mathematics. However, the conjecture seems to be hiding some property or some relation that involves powers of 2 and 3, and the methods that could be used to prove it, assuming it is provable, could have important applications in Mathematics, or, at least, they could provide us with a better understanding of the nature of the numbers themselves.

One of the most important sources of documentation in this work comes from the professor of the University of Michigan Jeffrey C. Lagarias, one of the most important contributors to our knowledge about the Collatz conjecture. His great book *The 3x + 1 Problem* [1], serves as a great guide to get introduced into the problem. Lagarias has also compiled all the remarkable bibliography on the Collatz conjecture until 2009 in [2] and [3]. A good quick introduction on the problem, alongside with some interesting approaches, and also an important input in this work, comes from the online blog of Terence Tao [4], who is one of the most talented mathematicians today.

1.1.1 Personal motivation

My personal interest for this conjecture began two and a half years ago after finishing the subject on Combinatorics and Graph Theory of the Mathematics degree of the Universitat Politècnica de Catalunya. After learning about generating functions and how to play with them, I realized how the Collatz conjecture, a conjecture I had already known for long and had always found incredibly interesting, can be thought as one, and managed to arrive to a result discovered 25 years ago by Berg and Meinardus [12], which we will briefly discuss in section 7.2.

Despite not being able to go much further from there, I kept being fascinated by the problem[2], and

[1] The problem is believed to have been created in the early thirties, and, although it seems to have circulated by word of mouth by mathematicians of the University of Cambridge, no papers were on the topic were written until the early seventies

[2] Richard Lipton calls this kind of problems "Mathematical Diseases", which are problems that are easy to state, accessible (even to people with not too much mathematical background), and that have plenty of "solutions" (in other words, you can find plenty of papers written by people who clearly are not professional mathematicians claiming to have a proof). The Collatz problem satisfies these three requirements.

so I tried to tackle it from many different approaches, with none of them giving any good results. Having realized that the Collatz conjecture hides a relation about powers of 2 and powers of 3, I tried to relate it to a concept that uses such powers, in this case, to a problem you start to appreciate when learning to program recursive functions, which is the problem of the Towers of Hanoi, and a video from the YouTube channel 3blue1brown, which makes great content on Mathematics (and which I also recommend to anyone who likes Mathematics, independently of their age or knowledge), showed me its relation with Sierpinski's triangle.

Finally, studying Sierpinski's triangle and trying to relate it to the Collatz conjecture, lead me to create this problem for a programming contest of my faculty. For some beautiful coincidence, the discussion of the problem with one of the participating students gave me the idea from which I could start developing the main part of the work.

1.2 Structure

We will structure this work the following way:

In Section 2, we will give some basic definitions regarding the Collatz conjecture, like what a Collatz sequence is or what is the Syracuse function, alongside with the statement of the conjecture. After that, we will also give some results about the conjecture that can be found in the literature.

In Section 3, we will explain what is the relation of the Collatz Problem with the powers of 2 and 3. With this, using the idea of the game of the Towers of Hanoi, we will arrive, via a beautiful relation between the two concepts, to Sierpinski's triangle. Here, we will see how a simple question about its structure will lead us to an interesting result that can be linked to the reformulation of the weak Collatz conjecture given by Böhm and Sontacchi, which will be presented in section 4, alongside with a proof of equivalence of the statements of the two formulations.

Sections 5 and 6 will be the original and core part of the work. In Section 5 we will introduce a new interesting class of generating functions, which we will name algebraic fractals, alongside with an interesting specific case, which we will call self-similar algebraic fractals. We will give some properties regarding their generators, inner structure, analytic behaviour, and equivalence to some kind of tensors. After this, we will introduce the notion of integer and boolean fractals, two particular cases of algebraic fractals, and we will give a geometric interpretation for them.

In Section 6 we will give, by following a connection we will observe between Sections 3 and 4, and using the theory developed in Section 5, an equivalent reformulation of the weak Collatz conjecture in terms of algebraic fractals, or more precisely, of self-similar boolean fractals.

In Section 7 we will explore, although just the surface, some other interesting approaches. In the first subsection, 7.1, we will present an original and interesting experimental result regarding pseudocollatz sequences, although we will not give too much depth to it since that would be out of the scope of this work. In the following two, 7.2 and 7.3, we will show, respectively, an approach regarding equivalences of the Collatz conjecture in terms of generating functions satisfying some functional equations, and an approach using transcendence theory, which can give interesting information about the separation of powers of 2 and 3.

We will end this work by giving some conclusions in Section 8. Following the bibliography that comes after this last section, there will two appendices, one presenting some code used to generate images or to study the Collatz conjecture, and one with just descriptive information of how the iterated Collatz function evolves after a small number of iterations.

1.3 Acknowledgements

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2. Preliminaries

Before we start with the main part of the work, it will be necessary to define the main concepts regarding the Collatz conjecture, which will be the main scope of this section. Furthermore, we will discuss a little bit about the current status of the conjecture, which will be useful to improve a little bit our understanding of the problem.

2.1 The Collatz function

Definition 2.1. We will define the slow Collatz function $C_s(n)$ as

$$C_s(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases} .$$

The Collatz conjecture is a pretty popular problem, and usually this is the definition used in outreach for its simplicity. However, we can note the following: For any odd integer n, $C(n) = 3n+1$ will be even. Although we are going a little bit ahead of ourselves, as the Collatz problem talks about the behaviour of the iterations of this function, it will more proper to define from now on the Collatz function the following manner, which jumps this unnecessary step in case of having and odd integer:

Definition 2.2. We will define the Collatz function $C(n)$ as

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases} .$$

As we want to know about what happens when we iterate this function many times, in would be convenient to have a way of denoting the iterated composition of a function.

Notation. Given any function $F : A \to A$, for some non-empty set A, we will denote as F^k, for any $k \in \mathbb{Z}_{\geq 0}$, its k-th fold composition, namely, the function obtained as a result of composing F a total of k times:

$$F^k(n) = \begin{cases} n & \text{if } k = 0 \\ F\left(F^{k-1}(n)\right) & \text{if } k > 0 \end{cases}$$

Observation 2.3. As it is immediate to see,

$$C(n) = \begin{cases} C_s(n) & \text{if } n \equiv 0 \pmod 2 \\ C_s^2(n) & \text{if } n \equiv 1 \pmod 2 \end{cases} .$$

This way, we are ready to define what iteration sequences are:

Definition 2.4. Given any function $F : A \to A$, for some non-empty set A, and given $x \in A$, we will call iteration sequence of x under F the sequence defined by $(x, F(x), F^2(x), F^3(x), ...)$.

Definition 2.5. For any strictly positive integer n, we will define its Collatz sequence as the iteration sequence of n under the Collatz function C, namely, $(n, C(n), C^2(n), C^3(n), ...)$.

Remark 2.6. The Collatz sequence of 1 is completely cyclic: As $C(1) = 2$, $C^2(1) = 1$, the resulting sequence is $(1, 2, 1, 2, 1, ...)$. The same can be said about the Collatz sequence of 2.

Definition 2.7. We will say that a cycle in a Collatz sequence is non-trivial if it does not contain the cycle $(1, 2)$. Equivalently, we will say that the cycle $(1, 2)$ in a Collatz sequence is the trivial one.

Once we have made this definitions, we are ready to state the Collatz conjecture, which will be the main focus in this work:

Conjecture 2.8 (Collatz conjecture). *For every $n \in \mathbb{Z}_{>0}$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that $C^k(n) = 1$.*

Observation 2.9. If $C^k(n) = 1$, we have that $(C^{k+1}(n), C^{k+2}(n), C^{k+3}(n), ...) = (2, 1, 2, 1, 2, ...)$. In other words, the Collatz conjecture is true if and only if, for each, $n \in \mathbb{Z}_{>0}$, its Collatz sequence "falls" into the trivial cycle. Equivalently, it will be false if there is an $n \in \mathbb{Z}_{>0}$ for which its Collatz sequence either enters a non-trivial cycle or goes to infinity.

Observation 2.10. If we do not restrict ourselves to the strictly positive integers, we may find these other cycles: (0), (-1), $(-5, -7, -10)$ and $(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)$.

In a similar manner as we have done before, where we passed from the slow Collatz function to a "faster" one (in the sense that, if it arrives to 1, it will need a smaller or equal number of steps to get there), we can easily define a new function that will be even faster:

Definition 2.11. We will define the Syracuse function $S(n) : \mathbb{N}_{odd} \to \mathbb{N}_{odd}$, where $\mathbb{N}_{odd}$ is the set of all non-negative odd integers, as

$$S(n) = \frac{3n+1}{2^{\nu_{3n+1}}},$$

where ν_n is the maximum power of 2 that divides n.

Observation 2.12. It is obvious that the image of the Syracuse function lies within $\mathbb{N}_{odd}$. It is also easy to see that

$$S(n) = C^{\nu_{3n+1}}(n).$$

We can see that, as $3n + 1$ is always even, $\nu_{3n+1} \geq 1$, and so, the Syracuse function advances at least as fast as the Collatz function.

Observation 2.13. In terms of the Syracuse function, the only trivial cycle will be (1) (it is obvious that 1 is a fixed point under such function) if and only if the Collatz conjecture is true. So, we could redefine the Collatz conjecture in terms of the Syracuse function. It will be true if and only if $\forall n \in \mathbb{Z}_{>0}$, its iteration sequence under the Syracuse function converges to 1.

Finally, as it would be needed throughout the work, we will also find it convenient to define a way to express a number in some particular base.

Notation. Let m be an integer such that $m \geq 2$. We can write a non-negative integer n as $n = d_k m^k + ... + d_1 m^1 + d_0 m^0$ (its representation in base m), for some $k \geq 0$ and $d_0, ..., d_k$ satisfying $d_i \in \{0, ..., m-1\}$. We will denote this by $n = (d_k \cdots d_1 d_0)_m$.

One of the main heuristic arguments to support the Collatz conjecture is the following. Because of the chaotic and pseudorandom behavior of Collatz sequences, it can be expected that half of their elements are odd and half are even. In this case, the Collatz function will take the form $\frac{3n+1}{2}$ half of the times, and $\frac{n}{2}$ the other half. In other words, the Collatz function is expected to multiply its argument by a ratio of approximately $\frac{3}{2}$ half of the times and by a ratio of $\frac{1}{2}$, the other half. Therefore, the ratio by which the Collatz function multiplies its argument would be, in average, $\sqrt{\frac{3}{2} \times \frac{1}{2}} = \frac{\sqrt{3}}{2} < 1$. So, the average behavior of the Collatz sequence would be a geometric sequence of ratio $\frac{\sqrt{3}}{2}$, which would make that, for any starting value, after a sufficient number of iterations we would get a number smaller than the starting one, making it impossible to have a first counterexample.

However, better arguments than the heuristic have been made. In the following paragraphs we will cover some of them.

2.2 Current status

Quite a lot of effort has been put to look for counterexamples of the Collatz conjecture. Oliveira and Silva [7] showed, in 2009, using some computational methods, that all numbers up to $\approx 5.76 \times 10^{18}$ satisfy the Collatz conjecture. There is also a collaborative project to look for counterexamples lead by Roosendaal [14], which has found that all numbers up to 10^{20} satisfy the conjecture.

Garner [8] gave in 1981 a lower bound on how long would a cycle different than the trivial one have to be, given a number N for which all $n < N$ satisfy the Collatz conjecture. This lower bound was improved, twelve years later, by Eliahou [9], who showed that any non-trivial cycle would necessarily have a cycle length of at least 1.04×10^{10} and would contain at least 6.58×10^9 odd numbers.

It would also be interesting to remark that, in 2002, Krasikov and Lagarias [10] showed that, for sufficiently large values of N, the number of integers n such that $1 \leq n \leq N$ and such that satisfy the Collatz conjecture, is at least $N^{0.84}$.

One of the most important generalizations on the Collatz Problem, made by John Conway [11], talks about the undecidability of certain functions similar to Collatz's. To be more precise, if we let G be the family of functions $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$g(n) = \begin{cases} a_0 n + b_0 & \text{if } n \equiv 0 \pmod{m} \\ \vdots & \vdots \\ a_{m-1} n + b_{m-1} & \text{if } n \equiv m-1 \pmod{m} \end{cases}$$

and such that $a_0, b_0, \dots, a_{m-1}, b_{m-1}$ are rational values for which the image is always in $\mathbb{Z}_{>0}$, then the problem that asks if $\forall g \in G$, $\forall n \in \mathbb{Z}_{>0}$, $\exists k$ for which $g^k(n) = 1$ is algorithmically undecidable, that is, it is not possible to construct an algorithm that, given any input $g \in G$, outputs correctly a yes or no to the previous answer.

Before we move ahead to the main content of this work, it would be interesting to stop for a moment to try to understand the chaotic behaviour of the iterated Collatz function and the patterns that arise inside such chaos.

One of the more interesting visualizations would be the so-called Collatz tree, which is represented in Figure 1. This structure represents a directed graph where each node represents an integer and has a unique outgoing edge, which goes to node that represents its

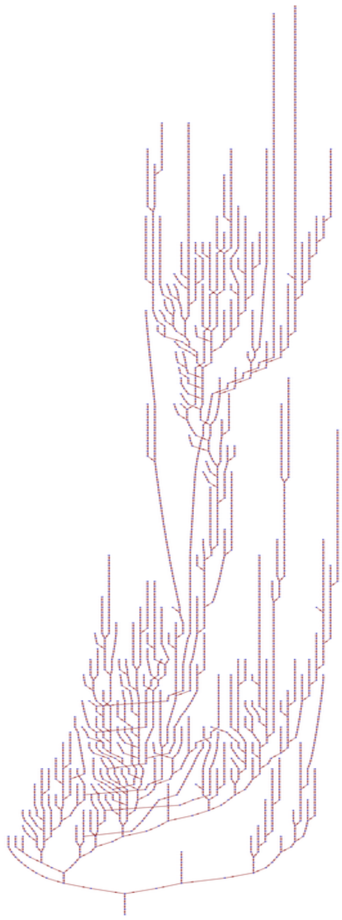

Figure 1: This Collatz tree represents the trajectories of the numbers from 1 to 1000 under the Collatz function. Its complicated structure may serve as a good idea of how complicated the behaviour of the iterations of the Collatz function really are.

image under the Collatz function. The structure of the graph is not exactly a tree, but it becomes a tree (assuming that in any Collatz sequence there are no non-trivial cycles) if the edge from 1 to 2 (the one that closes the trivial cycle) is removed. The Collatz conjecture is equivalent to stating that this graph is weakly connected, which means that, if we removed the "directedness" of the edges and convert the graph in an undirected graph, then this resulting graph is connected. At the end of this work we will give an analogous equivalent for this in terms of generating functions.

Another interesting visual would be the one represented in Figure 2, which plots, for a given integer, the number of iterations it takes to reach 1. As it is obvious, for a given number n, the minimum number of iterations will be $\lceil \log_2(n) \rceil$, in the best-case scenario for which all steps of the iteration are divisions by 2. As it can be seen in the image, both the lower part and upper part of the graph (that is, the curves that seem to bound the plot from the bottom and from the top), seems to present a logarithmic behaviour. The experimental data [14] (which can only be applied in the interval of numbers for which the Collatz conjecture has been tested), indicated that, by now, the largest constant C for which the number of steps $N(n)$ that the Collatz sequence of n needs to take to reach 1 is smaller or equal than $C \log(n)$, is $C \approx 36.72$. It seems plausible therefore to conjecture that $N(n) = \Theta(\log(n))$.

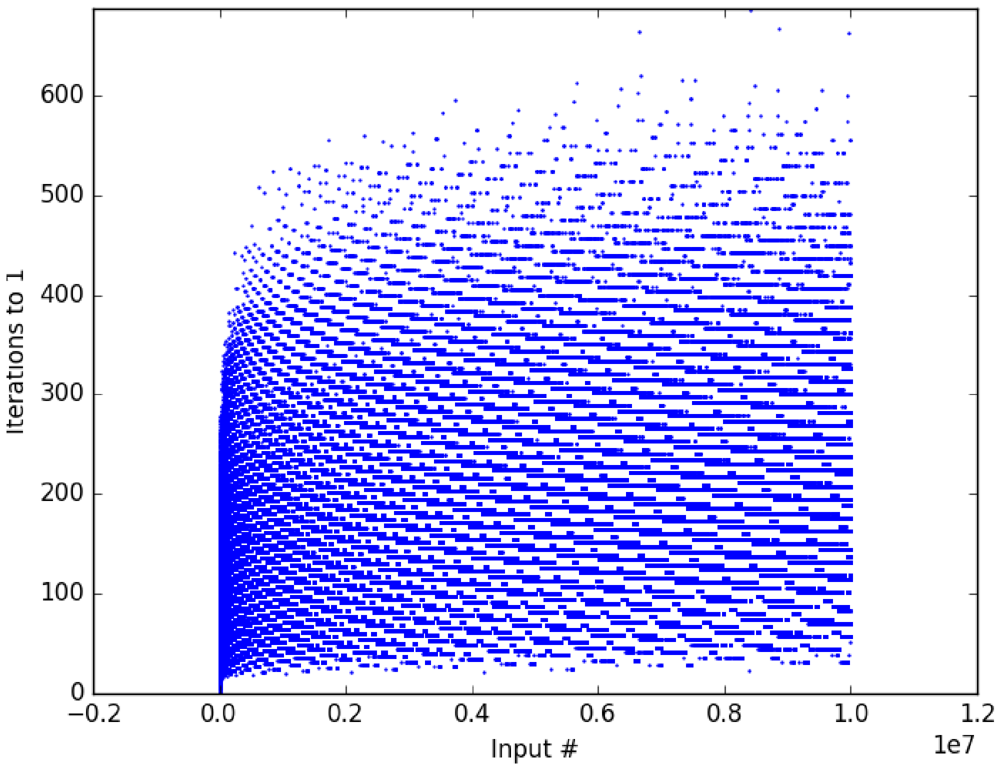

Figure 2: Plot of the number of iterations under the Collatz function needed to get to 1 depending on the initial value n.

3. Powers of 2 and powers of 3

We can easily see that we can rewrite the Collatz function as the function $C(n)$ for which $C(2n) = n$ and $C(2n+1) = 3n+2$. In other words, we can rethink the Collatz function just as two overlapped equations of a line, one of slope $\frac{1}{2}$ acting on the even naturals, and one of slope $\frac{3}{2}$ acting on the odd ones.

We can try, as an example, to see what happens when we compose the Collatz function with itself:

$$C^2(n) = C(C(n)) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{3n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{9n+5}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

which can be rewritten as $C^2(4n) = n$, $C^2(4n+1) = 3n+1$, $C^2(4n+2) = 3n+2$, $C^2(4n+3) = 9n+8$.

As we can see, in a similar way, we can rethink of $C^2(n)$ as 4 overlapped equations of a line, each acting on a subset of the positive integers with equal remainder modulo 4, one of them with slope $\frac{1}{4}$, one with slope $\frac{9}{4}$, and the other two with slope $\frac{3}{4}$. We can also observe that each of the two lines that made up $C(n)$ have subdivided in two, one multiplying its slope by $\frac{1}{2}$, and the other, by $\frac{3}{2}$. In Appendix B, it can be found a table of some other iterations $C^k(n)$ for small values of k, that may help to have a clue of the chaotic behaviour of the iterated Collatz function.

Once we have seen this example, we might try to see what happens to $C^k(n)$, for bigger values of k. This will lead us to the following theorem:

Theorem 3.1. *Let $k \in \mathbb{Z}_{\geq 0}$. We can write $C^k(x)$ as*

$$C^k(x) = \begin{cases} 3^{a_0^k} n + b_0^k & \text{if } x = 2^k n \\ 3^{a_1^k} n + b_1^k & \text{if } x = 2^k n + 1 \\ \vdots & \vdots \\ 3^{a_{2^k-1}^k} n + b_{2^k-1}^k & \text{if } x = 2^k n + (2^k - 1) \end{cases}.$$

Where a_j^k, b_j^k are positive integers. Furthermore, $0 \leq b_j^k < 3^{a_j^k}$, $0 \leq a_j^k \leq k$, and, $\forall r \in \{0, \dots, k\}$, $|\{j : a_j^k = r\}| = \binom{k}{r}$.

Proof. As $C^0(x) = x$, the statement trivially holds for $k = 0$. By definition of the Collatz function, and as $C^1(x) = C(x)$, the conjecture trivially holds for $k = 1$, since $C(2n) = n$ and $C(2n+1) = 3n+2$.

Now, we will proceed by induction. Assume the theorem is proven for $k-1$. For every $r \in \{0, \dots, 2^{k-1} - 1\}$ we will consider $C^k(2^{k-1} \cdot 2n + r)$ and $C^k(2^{k-1} \cdot (2n+1) + r)$. As we can rewrite these expressions as $C^k(2^k \cdot n + r)$ and $C^k(2^k \cdot n + (2^{k-1} + r))$, and as $\bigcup_{r \in \{0,\dots,2^{k-1}-1\}} \{r, r + 2^{k-1}\} = \{0, \dots, 2^k - 1\}$, we have a way of obtaining $C(2^k n + s), \forall s \in \{0, \dots, 2^k - 1\}$.

Now, on the one hand, $C^k(2^{k-1} \cdot 2n + r) = C(C^{k-1}(2^{k-1} \cdot 2n + r)) = C(3^{a_r^{k-1}} \cdot 2n + b_r^{k-1}) = C(2 \cdot (3^{a_r^{k-1}} n) + b_r^{k-1})$. On the other, $C^k(2^{k-1} \cdot (2n+1) + r) = C(C^{k-1}(2^{k-1} \cdot (2n+1) + r)) = C(3^{a_r^{k-1}} \cdot 2n + 3^{a_r^{k-1}} + b_r^{k-1}) = C(2 \cdot (3^{a_r^{k-1}} n) + 3^{a_r^{k-1}} + b_r^{k-1})$.

As it is clear, they differ in a term which is odd[3], which means that, after applying the Collatz function, one of the two choices r or $r + 2^{k-1}$, let us denote it by s, will be modified by the function $\frac{x}{2}$, giving a

[3]The fact that this difference is $3^{a_r^{k-1}}$ may give an intuitive idea of how chaotic the Collatz function can be.

new value of $3^{a_s^k}n + c_k$, with $a_s^k = a_r^{k-1}$, for some positive integer constant c_k, and the other, let us call it s', will be modified by the function $\frac{3x+1}{2}$, giving $3^{a_{s'}^k}n + c_k'$, with $a_{s'}^k = a_r^{k-1} + 1$, for some other positive integer constant c_k'. From this argument, we can deduce that

$$|\{j : a_j^k = r\}| = |\{j : a_j^{k-1} = r\}| + |\{j : a_j^{k-1} = r-1\}| = \binom{k-1}{r-1} + \binom{k-1}{r} = \binom{k}{r}.$$

In a similar way, we can see that the value M for which $a_M^k = \max_{i\in\{0,\dots,2^k-1\}} a_i^k$, will be either $a_M^{k-1}+1$ or $a_{M-2^{k-1}}^{k-1}+1$, which will mean, by induction, that, for the only value $M' \in \{M, M-2^{k-1}\}$ that satisfies $0 \le M' < 2^{k-1}-1\}$, $\forall i$, we will have $a_i^k \le a_M^k = a_{M'}^{k-1} + 1 \le k-1+1 = k$. The condition $a_i^k \ge 0$ is immediate.

The only thing left to prove is that $0 \le b_j^k < 3^{a_j^k}$. Now, let us compute $C^k(2^{k-1}\cdot 2n + r)$. We have $C^k(2^{k-1}\cdot 2n+r) = C(C^{k-1}(2^{k-1}\cdot 2n+r)) = C(3^{a_r^{k-1}}\cdot 2n + b_r^{k-1})$. We will distinguish between two cases:

If b_r^{k-1} is even:

$$C\left(3^{a_r^{k-1}}\cdot 2n + b_r^{k-1}\right) = C\left(2\left[3^{a_r^{k-1}}n + \frac{b_r^{k-1}}{2}\right]\right) = 3^{a_r^{k-1}}n + \frac{b_r^{k-1}}{2}$$

We see that $a_r^k = a_r^{k-1}$, $b_r^k = \frac{b_r^{k-1}}{2} \le b_r^{k-1} < 3^{a_r^{k-1}} = 3^{a_r^k}$.

If b_r^{k-1} is odd:

$$C\left(3^{a_r^{k-1}}\cdot 2n + b_r^{k-1}\right) = C\left(2\left[3^{a_r^{k-1}}n + \frac{b_r^{k-1}-1}{2}\right]+1\right) =$$

$$3\left[3^{a_r^{k-1}}n + \frac{b_r^{k-1}-1}{2}\right] + 2 = 3^{a_r^{k-1}+1}n + \left(3\frac{b_r^{k-1}-1}{2}+2\right)$$

We see that $a_r^k = a_r^{k-1}+1$ and $b_r^k = 3\frac{b_r^{k-1}-1}{2}+2$. Now:

$$3\frac{b_r^{k-1}-1}{2} + 2 = \frac{3b_r^{k-1}+1}{2} \le \frac{3\cdot(3^{a_r^{k-1}}-1)+1}{2} = \frac{3^{a_r^{k-1}+1}-2}{2} = \frac{3^{a_r^k}-2}{2} < 3^{a_r^k}$$

Now, let us compute $C^k(2^{k-1}\cdot(2n+1)+r)$. Again, we have $C^k(2^{k-1}\cdot(2n+1)+r) = C(C^{k-1}(2^{k-1}\cdot(2n+1)+r)) = C(3^{a_r^{k-1}}\cdot(2n+1) + b_r^{k-1})$.

In a similar way, if b_r^{k-1} is even:

$$C\left(3^{a_r^{k-1}}\cdot(2n+1) + b_r^{k-1}\right) = C\left(3^{a_r^{k-1}}\cdot 2n + 3^{a_r^{k-1}} + b_r^{k-1}\right) =$$

$$C\left(2\left[3^{a_r^{k-1}}n + \frac{3^{a_r^{k-1}}-1+b_r^{k-1}}{2}\right]+1\right) = 3\left[3^{a_r^{k-1}}n + \frac{3^{a_r^{k-1}}-1+b_r^{k-1}}{2}\right]+2 =$$

$$3^{a_r^{k-1}+1}n + \frac{3^{a_r^{k-1}+1}-3+3b_r^{k-1}}{2} + 2 = 3^{a_r^{k-1}+1}n + \frac{3^{a_r^{k-1}+1}+3b_r^{k-1}+1}{2}$$

We see that $a_r^k = a_r^{k-1}+1$ and $b_r^k = \frac{3^{a_r^{k-1}+1}+3b_r^{k-1}+1}{2}$. Now,

$$\frac{3^{a_r^{k-1}+1}+3b_r^{k-1}+1}{2} \le \frac{3^{a_r^{k-1}+1}+3(3^{a_r^{k-1}}-1)+1}{2} = 3^{a_r^k+1}-1 < 3^{a_r^k+1}$$

And, finally, if b_r^{k-1} is odd:

$$C\left(3^{a_r^{k-1}}\cdot(2n+1)+b_r^{k-1}\right)=C\left(3^{a_r^{k-1}}\cdot 2n+3^{a_r^{k-1}}+b_r^{k-1}\right)=$$

$$C\left(2\left[3^{a_r^{k-1}}n+\frac{3^{a_r^{k-1}}+b_r^{k-1}}{2}\right]\right)=3^{a_r^{k-1}}n+\frac{3^{a_r^{k-1}}+b_r^{k-1}}{2}$$

We see that $a_r^k=a_r^{k-1}$ and $b_r^k=\frac{3^{a_r^{k-1}}+b_r^{k-1}}{2}<\frac{3^{a_r^{k-1}}+3^{a_r^{k-1}}}{2}=3^{a_r^{k-1}}=3^{a_r^k}$. □

As an interesting consequence of this theorem, we may have this corollary, the proof of which can be found in [16]:

Corollary 3.2. *If we define $A(n)$ as $A(n)=\left|\left\{m\in\mathbb{Z}_{>0}: m<n\wedge\exists k \text{ such that } C^k(m)<m\right\}\right|$, then $\lim_{n\to\infty}\frac{A(n)}{n}=1$.*

3.1 Towers of Hanoi

As we can observe by analyzing the previous theorem, the seems to be hiding some relation between powers of 2 and powers of 3. Therefore, it may seem a good idea to relate the Collatz problem to some other mathematical concept that relates powers of 2 and powers of 3 as well. One of the first ideas that may come to someone's mind is the famous game of the towers of Hanoi.

Let us recall how this game works. The game consists of 3 rods, and n disks with different diameters. At the beginning, the leftmost rod has all the disks, sorted from smaller diameter (on the top) to larger (on the bottom). The goal of the game is to move the n disks to the rightmost rod, with just two restrictions. First, only one disk can be moved at a time and the only disks that can be moved are those at the top of some rod. Second, a disk of a certain diameter cannot be placed above a disk of smaller diameter.

It is clear that the game of the Towers of Hanoi with n disks has a total of 3^n possible configurations. Let us give an argument for why is this true:

It is clear that the disk of greater diameter must be at the bottom of one of the three rods. The second greatest disk can also be in any of the three rods, since its placement does not depend on the position of the greatest, since it can either be in some of the 2 empty rods or in the other rod, right above the first disk. In general, by the same argument, the position of a disk is independent of the position of the disks which are bigger than it, since placing them in any of the three rods does not violate the condition of not having a disk over one with smaller diameter. Therefore, we have 3 independent choices for each of the n disks, getting a total of 3^n configurations.

It is well-known that the minimum number of moves to solve the game $M(n)$, is $M(n)=2^n-1$. To move the biggest disk from the leftmost rod to the rightmost one it is necessary to have all the disks in the middle rod (otherwise, some small disk would be on the left rod, blocking the biggest one to being moved from there, or would be on the right rod, which would block it from landing there). In other words, the game of the towers of Hanoi with $n-1$ disks must have been solved using the middle rod as the final rod. After this, one must move the biggest disk to the right, and solve again the game for $n-1$ disks using the middle rod as the starting rod. This generates the recurrence $M(n)=M(n-1)+1+M(n-1)=2M(n-1)+1$. As $M(1)=1$, which is immediate to see, it is very easy to check by induction that $M(n)=2^n-1$. In other words, the optimal solution will go through 2^n different states.

The graph of states of the game of Towers of Hanoi is a graph having each possible configuration of the game as a node, and an edge between two nodes if it is possible change from the state of one of the nodes to the state of the other by performing a single move. As all the moves are reversible, the graph is undirected. This graph has a very nice property: its natural representation has the shape of a Sierpinski Triangle, as we can see in Figure 3.

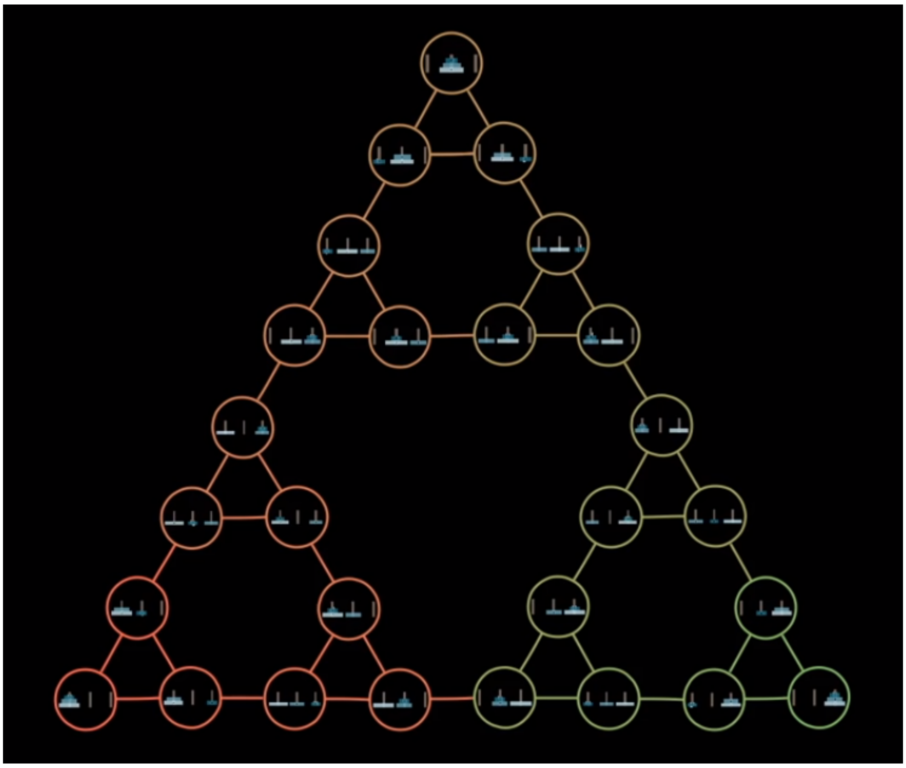

Figure 3: Graph of states of the Towers of Hanoi. The initial state is located in the lower left corner, whereas the final one is in the lower right one. The graph has 2^n states in each side and a total of 3^n states. The chain of states at the bottom side, which, as is not difficult to see, is the shortest path between the first and last states, correspond to the sequence of moves to solve the puzzle in the minimum possible number of moves. As it goes through 2^n states, it makes $2^n - 1$ moves, as we commented previously.

3.2 Sierpinski's triangle

Let us start with a simple question. How many small triangles have the first n rows of the Sierpinski triangle?[4] Let us make it clear, since the Sierpinski's triangle can be defined in many ways, that we are using the case shown in Figure 4.

The following two lemmas will help us answer this question.

Lemma 3.3. *For every prime p and integer $m \geq 0$, the following relation holds:* $(1+x)^{p^m} \equiv 1 + x^{p^m} \pmod{p}$.

[4] This question has been (accidentally) inspired by the problem *Pascal al Louvre* from the 15th Programming Contest on Algorithms of the Faculty of Mathematics and Statistics of the Universitat Politècnica de Catalunya. A link to the problem can be found here.

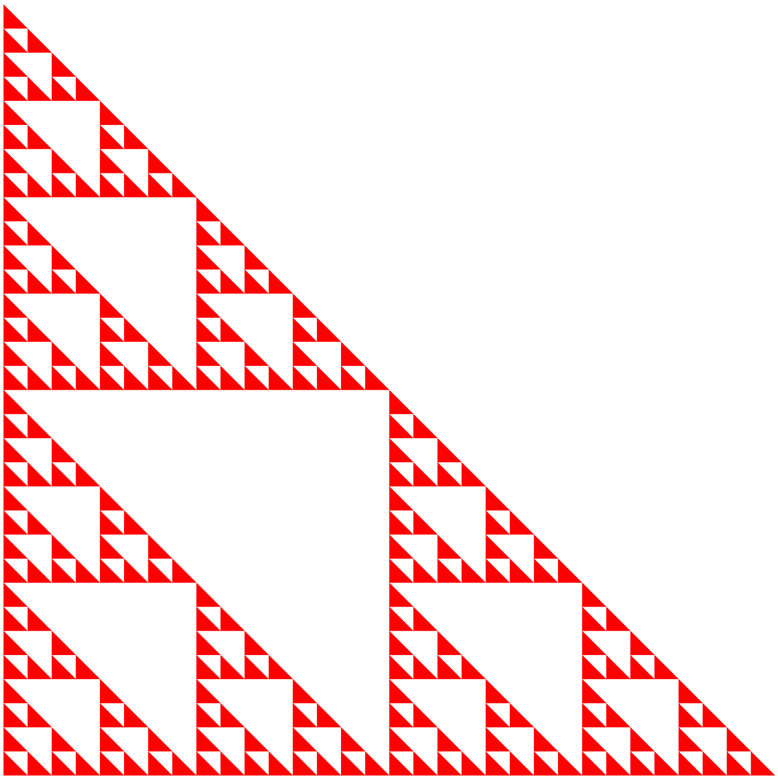

Figure 4: The 32 first rows of Sierpinski's triangle.

Proof. Let us prove the statement by induction. For $m = 0$ it trivially holds. For $m = 1$,

$$(1+x)^{p^1} = (1+x)^p = \sum_{j=0}^{p} \binom{p}{j} x^j = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!} x^j$$

We see that, as p is prime, p will divide $j!$ if and only if $j \geq p$ (which will only happen at $j = p$), and similarly, p will divide $(p-j)!$ if and only if $p - j \geq p$ (which will only happen at $j = 0$). Therefore, $\frac{p!}{j!(p-j)!} \equiv 0 \pmod{p}$ if $0 < j < p$, and so:

$$\sum_{j=0}^{p} \frac{p!}{j!(p-j)!} x^j \equiv \frac{p!}{0!p!} x^0 + \frac{p!}{p!0!} x^p \equiv 1 + x^p \pmod{p}.$$

Now, we can proceed by induction. If the statement is true for $m - 1$:

$$(1+x)^{p^m} = (1+x)^{p^{m-1}p} = \left((1+x)^{p^{m-1}}\right)^p \equiv \left(1 + \left(x^{p^{m-1}}\right)\right)^p \equiv 1 + \left(x^{p^{m-1}}\right)^p \equiv 1 + x^{p^m} \pmod{p}$$

□

Lemma 3.4. *The n-th row of Sierpinski's triangle (where the starting row will be the 0-th row) has 2^{d_n} triangles, where d_n is the number of non-zero bits of n when it is expressed in base 2.*

Proof. The number of of triangles of the n-th row is equal to the number of elements of the set $S_n = \{k : 0 \le k \le n \wedge \binom{n}{k} \equiv 0 \pmod 2\}$. As $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}$, $|S_n|$ will be the number of odd coefficients of $(1+x)^n$, or equivalently, the number of non-zero coefficients of $(1+x)^n$ taken modulo 2. Now, if the base 2 expression of n is $n = (b_r b_{r-1}\cdots b_1 b_0)_2$, and $i_1 < \ldots < i_{d_n}$ are the indices such that b_{i_j} will be non-zero, we will have $n = 2^{i_{d_n}} + \ldots + 2^{i_0} = \sum_{j=0}^{d_n} 2^{i_j}$. Now, using the previous lemma with $p = 2$, we obtain the following:

$$(1+x)^n = (1+x)^{\sum_{j=1}^{d_n} 2^{i_j}} = \prod_{j=1}^{d_k}(1+x)^{2^{i_j}} \equiv \prod_{j=1}^{d_k}\left(1+x^{2^{i_j}}\right) \pmod 2$$

If we expand this product, we get 2^{d_k} different summands each of which are monomials of the form x^m, meaning $(1+x)^n$ modulo 2 has 2^{d_n} non-zero coefficients, as we wanted to see. □

As a result of these lemmas, we can give an alternative proof for the following property:

Corollary 3.5. *The first 2^n rows of Sierpinski's triangle contain 3^n triangles.*

Proof. Let $0 \le k \le n$. There are $\binom{n}{k}$ different strings of bits of length n having k zeroes. The numbers from 0 to $2^n - 1$ can be expressed as the 2^n possible different strings of bits of length n. As there are $\binom{n}{k}$ of these made up of k 1's, the total number of small triangles will be $\sum_{k=0}^{n}\binom{n}{k}2^k$. Using that $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^n$, plugging $x = 2$, we get $\sum_{k=0}^{n}\binom{n}{k}2^k = (1+2)^n = 3^n$. □

Once we have proven these lemmas, we can make the following observation: In base two, each positive integer n can be expressed as $n = (b_r b_{r-1}\cdots b_1 b_0)_2$, with $b_i \in \{0,1\}$. Let $0 \le a_0 < \ldots < \ldots < a_k = r$ be the indices of the non-zero bits, that is, $b_{a_i} = 1$ for every i, and $b_j = 0$ if there is not i such that $a_i = j$. We can now rewrite n as $n = 2^{a_k} + 2^{a_{k-1}} + \ldots + 2^{a_1} + 2^{a_0} = \sum_{j=0}^{k} 2^{a_j}$.

For the rest of the section, we will index the rows starting by 1. Now, let $s(n)$ be the number of triangles between the first row and row number n (both included). If, $n = 2^k$ for some $k \in \mathbb{Z}_{\ge 0}$, by Theorem 3.5, we will have $s(n) = s(2^k) = 3^k$. Otherwise, we will have $2^k < n < 2^{k+1}$, for some $k \in \mathbb{Z}_{\ge 0}$. By the own structure of the Sierpinski's Triangle, the triangles between the row $2^k + 1$ and the row n (both included) are, by symmetry, twice the number of triangles between the same rows but just on the left half, which at its turn is equal, by construction of the Sierpinski triangle, to the number of triangles between the row 1 and the row $n - 2^k$ (both included), as we can see in Figure 4. We therefore have, in the case $2^k < n < 2^{k+1}$, the following recurrence:

$$s(n) = s(2^k) + 2s(n - 2^k).$$

Clearly, we will have that 2^k and $n - 2^k$ are both positive integers which are strictly less than n. However, again by Theorem 3.5, we have that $\forall k \in \mathbb{Z}_{\ge 0}$, $s(2^k) = 3^k$. Therefore, we can rewrite the recurrence $s(n)$ as:

$$s(n) = \begin{cases} 3^k & \text{if } n = 2^k \\ s(2^k) + 2s(n-2^k) & \text{if } 2^k < n < 2^{k+1} \end{cases}$$

We are now ready to prove the following theorem:

Theorem 3.6. *Let n be an integer that can be expressed as $n = 2^{a_k} + \ldots + 2^{a_0}$, for some integers $k \in \mathbb{Z}_{\ge 0}$ and $a_0, \ldots, a_k$ such that $0 \le a_0 < a_1 < \cdots < a_k$. Then $s(n) = \sum_{j=0}^{k} 3^{a_j} 2^{k-j}$.*

Proof. We will prove this formula by induction on k. If $k = 0$, $n = 2^{a_0}$ and so, $s(n) = s(2^{a_0}) = 3^{a_0}$, which satisfies the formula.

Assume the theorem is proven up to $k - 1$. Now, if $n = 2^{a_k} + 2^{a_{k-1}} + ... + 2^{a_0}$, we clearly have $2^{a_k} < n < 2^{a_k+1}$, and so:

$$s(n) = s(2^{a_k}) + 2s(n - 2^{a_k}) = 3^{a_k} + 2s(2^{a_k} + 2^{a_{k-1}} + ... + 2^{a_0} - 2^{a_k}) =$$

$$3^{a_k} + 2s(2^{a_{k-1}} + ... + 2^{a_0}) = 3^{a_k} + 2\sum_{j=0}^{k-1} 3^{a_j}2^{k-1-j} = 3^{a_k} + \sum_{j=0}^{k-1} 3^{a_j}2^{k-j} = \sum_{j=0}^{k} 3^{a_j}2^{k-j}$$

□

At the end of this work, we are going to see why this theorem will be an important inspiration to come up with the main result of this work.

4. The weak Collatz conjecture

Let us revisit again what the Collatz conjecture actually states. Assuming it was true, the fact that the Collatz sequence eventually reaches 1, means, on the one hand, that there is not a positive integer such that its Collatz sequence goes to infinity, and on the other, that there is not a positive integer such that its Collatz sequence enters a non-trivial cycle. In other words, the proof of the conjecture may not be just one single proof, but the proof of two different conjectures, one stating that no Collatz sequence goes towards infinity, and one stating that there are no non-trivial cycles.

We will try to focus on the latter case. To such purpose, let us state the weak Collatz conjecture:

Conjecture 4.1 (Weak Collatz conjecture)**.** *For any $n \in \mathbb{Z}_{>0}$, if its Collatz sequence has a cycle, then such cycle is trivial.*

In 1978, Böhm and Sontacchi [6] managed to give an equivalent formulation of the conjecture, which talks about some divisibility property regarding a combination of powers of 2 and powers of 3.

Conjecture 4.2 (Weak Collatz conjecture, reformulated)**.** *There does not exist integers $k \geq 0$, and $a_0, a_1, \ldots, a_k, b$ such that $0 = a_0 < a_1 < \ldots < a_k < b$ for which*

$$\left(2^b - 3^{k+1}\right) n = \sum_{j=0}^{k} 3^{k-j} 2^{a_j}$$

for some integer $n > 1$.

It is not easy to see that both conjectures are, indeed, equivalent. The goal of this section is to state the two formulations of the weak Collatz conjecture and to prove their equivalence, which we are going to do next. The following proof of equivalence is based on the one given by Terence Tao in an article on his blog about the Collatz conjecture [4]:

Proposition 4.3. *The two previous conjectures are, indeed, equivalent.*

Proof. Let us define the following equivalence relation $\sim$ on $\mathbb{Z}_{\geq 1}$. We will say that $a \sim b$ if and only if exists $m \in \mathbb{Z}$ such that $\frac{a}{b} = 2^m$. It is easy to see that, indeed, this is an equivalence relation: $a \sim a$ since $\frac{a}{a} = 2^0$, if $a \sim b$, which means $\frac{a}{b} = 2^m$, then $\frac{b}{a} = 2^{-m}$, meaning $b \sim a$, and finally, if $a \sim b$ and $b \sim c$, meaning $\frac{a}{b} = 2^{m_1}$ and $\frac{b}{c} = 2^{m_2}$ for some integers m_1, m_2, then $\frac{a}{c} = \frac{a}{b}\frac{b}{c} = 2^{m_1+m_2}$.

Now, let us define[5] $\tilde{C} : (\mathbb{Z}_{\geq 1}/\sim) \to (\mathbb{Z}_{\geq 1}/\sim)$ such that $\tilde{C}([n]) = [3n + 2^{\nu_n}]$, where ν_n will be the integer $\nu_n = \max\left\{k \in \mathbb{Z}_{\geq 0} : 2^k | n\right\}$. It is easy to see that the function is well defined. If $[n] = [n']$, then $\frac{n'}{n} = 2^m$ for some $m \in \mathbb{Z}$. Clearly, $\nu_{n'} = \nu_n + m$. Therefore, $\frac{3n'+2^{\nu_{n'}}}{3n+2^{\nu_n}} = \frac{3n \times 2^m + 2^{\nu_n + m}}{3n+2^{\nu_n}} = 2^m$, and so, $\tilde{C}([n])$ is well-defined.

Let us show the following by induction: If, again, we denote by $\tilde{C}^k$ the k-fold composition of the function $\tilde{C}$, if n is an odd number, then, $\tilde{C}^k([n]) = [3^k n + 3^{k-1} 2^{a_0} + 3^{k-2} 2^{a_1} + \ldots + 2^{a_{k-1}}]$, for some $0 = a_0 < a_1 < \cdots < a_{k-1}$ and such that, $a_i = \nu_{3^i n + 3^{i-1} 2^{a_0} + 3^{i-2} 2^{a_1} + \ldots + 2^{a_{i-1}}}$.

Let us show this. For $k = 0$ and $k = 1$ the statement trivially holds.

[5] It is easy to see that the function $\tilde{C}$ is essentially the Syracuse function: as every equivalence class has exactly one odd number (which is trivial to see), it is easy to see that, if m is the only odd number such that $m \in [n]$, and m' is the only odd number such that $m' \in \tilde{C}([n])$, then $S(m) = m'$, where $S(n)$ is the Syracuse function.

Assuming the statement is true for $k-1$, then

$$\tilde{C}^k([n]) = \tilde{C}(\tilde{C}^{k-1}([n])) = \tilde{C}([3^{k-1}n + 3^{k-2}2^{a_0} + ... + 2^{a_{k-2}}]) = [3(3^{k-1}n + 3^{k-2}2^{a_0} + ... + 2^{a_{k-2}}) + 2^{a_{k-1}}]$$

Let $n_{k-1} = 3^{k-1}n + 3^{k-2}2^{a_0} + ... + 2^{a_{k-2}}$. By induction hypothesis, $\nu_{n_{k-1}} = 2^{a_{k-1}}$. Therefore, $3(3^{k-1}n + 3^{k-2}2^{a_0} + ... + 2^{a_{k-2}}) + 2^{a_{k-1}} = 3m_{k-1}2^{a_{k-1}} + 2^{a_{k-1}} = (3m_{k-1}+1)2^{a_{k-1}}$, for some odd integer m_{k-1}. Since $3m_{k-1}+1$ is even, this means that $a_k = \nu_{3(3^{k-1}n+3^{k-2}2^{a_0}+...+2^{a_{k-2}})+2^{a_{k-1}}} = \nu_{(3m_{k-1}+1)2^{a_{k-1}}} > a_{k-1}$. Finally,

$$[3(3^{k-1}n + 3^{k-2}2^{a_0} + ... + 2^{a_{k-2}}) + 2^{a_{k-1}}] = [3^k n + 3^{k-1}2^{a_0} + ... + 3 \times 2^{a_{k-2}} + 2^{a_{k-1}}]$$

as we wanted to see, proving the statement $\forall k \in \mathbb{Z}_{\geq 0}$

Now, let us assume that Collatz conjecture has some non-trivial cycle. This is equivalent to saying that, for some $[n] \neq [1]$, (assume n is odd), there is some k for which $\tilde{C}^{k+1}([n]) = [n]$. In this case, there would be an integer b such that

$$2^b n = 3^{k+1}n + 3^k 2^{a_0} + ... + 3 \cdot 2^{a_{k-1}} + 2^{a_k}$$

Since n is odd and the RHS is an integer, b cannot be a non-negative integer. Using the same reasoning we used to see that $a_0 < \cdots < a_k$, we can see that $b > a_k$. Rearranging the above equation, we get to:

$$(2^b - 3^{k+1})n = 3^k 2^{a_0} + ... + 3 \cdot 2^{a_{k-1}} + 2^{a_k}$$

Proving the equivalence. □

Observation 4.4. By the own construction of the proof, b will be the number of $\frac{x}{2}$ steps, under the slow Collatz function, needed to get back to the initial odd number n, which will be an element of a non-trivial cycle, and $k+1$ will be the number of $3x+1$ steps taken until n cycles back to itself. Equivalently, b and $k+1$ will be, respectively, the total number of steps and the number of $\frac{3x+1}{2}$ steps taken under the Collatz function until n cycles back to itself.

Finally, it would be interesting to see the following corollary, which gives us a very tight bound on what the value b can be:

Corollary 4.5. *Let $N > 1$ be an integer such that $\forall m < N$, the Collatz sequence starting at m gets to 1. Assume that a counterexample for the previous conjecture is true, which would mean that there exists $0 = a_0 < a_1 < ... < a_k < b$ and $n > 1$ such that $(2^b - 3^{k+1})n = \sum_{j=0}^{k} 3^{k-j}2^{a_j}$. Then the following is true:*

$$\frac{\log 3}{\log 2}(k+1) < b < \frac{\log\left(3 + \frac{1}{N}\right)}{\log 2}(k+1)$$

Proof. The leftmost inequality is equivalent to $3^{k+1} < 2^b$, which is trivially true since $2^b - 3^{k+1}$ is positive.

Let us prove the other inequality. If $r \geq N > 1$ is the first number for which the Collatz sequence non-trivially cycles back to itself, on the one hand, by the previous observation, the Collatz function will provide $(k+1)$ steps in which the values are changed by the function $\frac{3n+1}{2}$, and on the other, $C^t(r) \geq r \ \forall t \in \mathbb{Z}_{>0}$, since, otherwise, if $C^t(r) = q < r$, as there is an integer s such that $C^s(r) = r$, we would have $C^s(q) = C^s(C^t(r)) = C^{s+t}(r) = C^t(C^s(r)) = C^t(r) = q$, which would contradict that r is the first integer that non-trivially cycles under the Collatz sequence.

Therefore, the $k+1$ transformations of type $\frac{3n+1}{2}$ will be performed on some integers t_i of the Collatz sequence, all greater or equal than r (which, in turn, is bigger or equal than N), for which we would get

$\frac{3t_i+1}{2} = \frac{3+\frac{1}{t_i}}{2} t_i \leq \frac{3+\frac{1}{N}}{2} t_i$. By the previous observation, b will be the number of steps that the Collatz function will need to take to close the cycle. Therefore, $k+1$ out of the b transformations will yield to a result smaller or equal than multiplying their argument by a factor of $\frac{3+\frac{1}{N}}{2}$, whereas the other $b-(k+1)$ will be just multiplying by $\frac{1}{2}$. So, as after these b steps the function will cycle back to itself, we would get

$$1 \leq \left(\frac{3+\frac{1}{N}}{2}\right)^{k+1} \left(\frac{1}{2}\right)^{b-(k+1)} = \left(3+\frac{1}{N}\right)^{k+1} \left(\frac{1}{2}\right)^{b},$$

leading to $2^b \leq \left(3+\frac{1}{N}\right)^{k+1}$, or, equivalently, by taking logarithms, to $b \leq \frac{\log\left(3+\frac{1}{N}\right)}{\log 2}(k+1)$. Finally, we show that the inequality

$$b < \frac{\log\left(3+\frac{1}{N}\right)}{\log 2}(k+1)$$

is strict. Indeed, equality is not possible since the LHS is an integer whereas the RHS is irrational (and actually, transcendental, which can be proven using Baker's theorem, which we will talk about at the end of this work). Let us show this last point. If $\frac{\log\left(3+\frac{1}{N}\right)}{\log 2}$ is rational, $\frac{\log\left(3+\frac{1}{N}\right)}{\log 2} = \frac{p}{q}$, for some integers p and q, leading to $q\log\left(3+\frac{1}{N}\right) = p\log 2$, which means $(3+\frac{1}{N})^q = 2^p$, leading to a contradiction since the RHS is an integer but the LHS is not since $N > 1$. $\square$

5. Relation with fractals

5.1 Algebraic fractals

In this section, we will start by giving the first original contribution of this work, which will be the introduction of the concept of algebraic fractals, and, from there, we will be able to define what a self-similar algebraic fractal is, and what integer and boolean fractals are, alongside with their geometric interpretation.

Recall that $\mathbb{C}[[z]]$ is the ring of formal power series in the variable z over $\mathbb{C}$, that is, the set of power series $\sum_{k=0}^{\infty} a_k z^k$, where $a_k \in \mathbb{C}\ \forall k \in \mathbb{Z}_{\geq 0}$. So, let us start by defining what an algebraic fractal is:

Definition 5.1. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$, $P(z) = \sum_{k=0}^{\infty} p_k z^k \in \mathbb{C}[[z]]$ and $m \in \mathbb{Z}$, with $p_0 = 1$ and $m \geq 2$. We will say that $A(z)$ is an algebraic fractal generated by $P(z)$ with order m if and only if we can express $A(z)$ as:

$$A(z) = P(z)P(z^m)P\left(z^{m^2}\right)\cdots = \prod_{r=0}^{\infty} P\left(z^{m^r}\right).$$

Observation 5.2. We may observe that no generator $P(z)$ of an algebraic fractal $A(z)$ is unique (except if $A(z) \equiv 1$). If $P(z)$ has order m, $Q(z) = P(z)P(z^m) \neq P(z)$ will be a different generator of $A(z)$, but having order m^2, since

$$\prod_{r=0}^{\infty} Q\left(z^{(m^2)^r}\right) = \prod_{r=0}^{\infty} P\left(z^{(m^2)^r}\right) P\left((z^m)^{(m^2)^r}\right) = \prod_{r=0}^{\infty} P\left(z^{m^{2r}}\right) P\left(z^{m^{2r+1}}\right) = \prod_{s=0}^{\infty} P\left(z^{m^s}\right) = A(z).$$

With a similar reasoning, we can see that, $\forall s \in \mathbb{Z}_{>0}$, $\prod_{r=0}^{s-1} P\left(z^{m^r}\right)$ is a generator of order m^s, giving us a way of finding infinitely many different generators for the same algebraic fractal.

The following proposition is immediate:

Proposition 5.3. *Let $m \geq 2$ be an integer. Now, let $A(z) \in \mathbb{C}[[z]]$ be an algebraic fractal generated by $P(z) \in \mathbb{C}[[z]]$ with order m and let $B(z) \in \mathbb{C}[[z]]$ be an algebraic fractal generated by $Q(z) \in \mathbb{C}[[z]]$ with order m. Then $C(z) = A(z)B(z)$ is an algebraic fractal generated by $R(z) = P(z)Q(z)$ with order m.*

Proof.

$$\prod_{r=0}^{\infty} R\left(z^{m^r}\right) = \prod_{r=0}^{\infty} P\left(z^{m^r}\right) Q\left(z^{m^r}\right) = \left(\prod_{r=0}^{\infty} P\left(z^{m^r}\right)\right)\left(\prod_{r=0}^{\infty} Q\left(z^{m^r}\right)\right) = A(z)B(z) = C(z).$$

□

The following definition will be useful for the developing of this section:

Definition 5.4. Let $Q(z) = \sum_{k=0}^{\infty} q_k z^k \in \mathbb{C}[[z]]$, and let $r \in \mathbb{Z}_{\geq 0}$. We will define $Q_{[r]}(z)$ as $Q_{[r]}(z) = \sum_{k=0}^{r-1} q_k z^k$.

Now, let $n \in \mathbb{Z}_{>0}$. Let us define the following equivalence relation on $\mathbb{C}[[z]]$: We will say $P(z) \sim_n Q(z)$ if and only if $P_{[n]}(z) = Q_{[n]}(z)$ (it is trivial to check that, indeed, it is an equivalence relation). In the quotient set $\mathbb{C}[[z]]/\sim_n$, if we denote the equivalence class of $P(z)$ under this equivalence relation as

$[P(z)]_n$, we can define a sum and a product by $[P(z)]_n + [Q(z)]_n = [P(z) + Q(z)]_n$ and $[P(z)]_n[Q(z)]_n = [P(z)Q(z)]_n$. Let us see that the sum and the product are well defined:

Let $P(z) = \sum_{k=0}^{\infty} p_k z^k$, $Q(z) = \sum_{k=0}^{\infty} q_k z^k$. In the case of the sum, we would clearly have $[P(z)]_n + [Q(z)]_n = [P(z) + Q(z)]_n = \sum_{k=0}^{n-1}(p_k + q_k)z^k$. In the case of the multiplication, we would have $[P(z)]_n[Q(z)]_n = [P(z)Q(z)]_n = [R(z)]_n = \sum_{k=0}^{n-1} r_k z^k$, where $R(z) = P(z)Q(z) = \sum_{k=0}^{\infty} r_k z^k$, with $r_k = \sum_{s=0}^{k} p_s q_{k-s}$. As we can see, $\forall k < n$, r_k only depends on $p_0, \ldots, p_{n-1}, q_0, \ldots, q_{n-1}$.

As we have seen, both $[P(z)]_n + [Q(z)]_n$ and $[P(z)]_n[Q(z)]_n$ will only depend on the first n coefficients of $P(z)$ and the first n coefficients of $Q(z)$, namely, $p_0, \ldots, p_{n-1}, q_0, \ldots, q_{n-1}$. So, if $\hat{P}(z) \in [P(z)]_n$, $\hat{Q}(z) \in [Q(z)]_n$, we would have $\hat{P}(z) = \sum_{k=0}^{n-1} p_k z^k + \sum_{n}^{\infty} \hat{p}_k z^k$, $\hat{Q}(z) = \sum_{k=0}^{n-1} q_k z^k + \sum_{n}^{\infty} \hat{q}_k z^k$, meaning $[P(z)]_n + [Q(z)]_n = [\hat{P}(z)]_n + [\hat{Q}(z)]_n$ and $[P(z)]_n[Q(z)]_n = [\hat{P}(z)]_n[\hat{Q}(z)]_n$, which proves that the such sum and product are well-defined.

In a similar way, we can define an scalar multiplication in this quotient set, by saying that, if $a \in \mathbb{C}$, then $a[P(z)]_n = [aP(z)]_n$, which is trivially well-defined as well.

Proposition 5.5. *There exists a natural bijection $f : (\mathbb{C}[[z]]/\sim_n) \to \mathbb{C}_{[n]}[z]$, where $\mathbb{C}_{[n]}[z] \subset \mathbb{C}[[z]]$ denotes the set of complex polynomials of degree $< n$, given by $f([P(z)]_n) = P_{[n]}(z)$, with inverse $g = f^{-1}$ given by $g(P(z)) = [P(z)]_n$.*

Proof. It is clear that, on the one hand, f is well-defined, and on the other $P_{[n]}(z) \in [P(z)]_n$. Now, $g(f([P(z)]_n) = g(P_{[n]}(z)) = [P_{[n]}(z)]_n = [P(z)]_n$, and, at the same time, if $P(z)$ is a polynomial of degree $< n$, we have $P_{[n]}(z) = P(z)$, and so, $f(g(P(z))) = f([P(z)]_n) = P_{[n]}(z) = P(z)$. □

Notation. Let $F(z) = \sum_{k=0}^{\infty} f_k z^k \in \mathbb{C}[[z]]$. We will say that $[z^k]F(z)$ is the k-th coefficient of $F(z)$, namely $[z^k]F(z) = f_k$.

Now, it will be useful to know, for the rest of the section, when a power series has a multiplicative inverse:

Proposition 5.6. *Let $F(z) = \sum_{k=0}^{\infty} f_k z^k \in \mathbb{C}[[z]]$, then, $F(z)$ has a unique multiplicative inverse $G(z) = \sum_{k=0}^{\infty} g_k z^k \in \mathbb{C}[[z]]$ (in other words, a power series with complex coefficients that satisfies $F(z)G(z) = 1$) if and only if $f_0 \neq 0$. Furthermore, the coefficients of $G(z)$ can be defined recursively by setting $g_0 = \frac{1}{f_0}$, and, for $k > 0$, $g_k = -\frac{1}{f_0}\sum_{i=0}^{n-1} f_{n-i}g_i$.*

Proof. It is clear that there does not exist such inverse if $f_0 = 0$ since we would have $1 = [z^0](1) = [z^0](F(z)G(z)) = f_0 g_0 = 0$, which is a contradiction. Otherwise, we would have, using such construction of $G(z)$, $[z^0](F(z)G(z)) = f_0 g_0 = 1$, and, if $k > 0$,

$$[z^k](F(z)G(z)) = \sum_{i=0}^{n} f_{n-i}g_i = \sum_{i=0}^{n-1} f_{n-i}g_i + f_0 g_n = \sum_{i=0}^{n-1} f_{n-i}g_i + f_0\left(-\frac{1}{f_0}\sum_{i=0}^{n-1} f_{n-i}g_i\right) = 0,$$

meaning that $G(z)$ is indeed a multiplicative inverse of $F(z)$. Finally, uniqueness is easy to prove. Assume $H(z) \in \mathbb{C}[[z]]$ is another multiplicative inverse of $F(z)$, which, by definition, would mean $1 = F(z)H(z)$. Then, multiplying both sides by $G(z)$ we would get $G(z) = G(z)F(z)H(z) = (G(z)F(z))H(z) = H(z)$, getting a contradiction and completing the proof. □

Once we have defined what algebraic fractals are, and once we have seen the previous proposition and the equivalence class $\sim_n$, we are ready to give some properties about algebraic fractals:

Proposition 5.7. *Algebraic fractals satisfy the following properties:*

1. *For any algebraic fractal $A(z)$ with a generator $P(z)$ of order m, we have $a_i = p_i$, $\forall i$ such that $0 \leq i < m$.*
2. *If the algebraic fractal $A(z)$ is generated by $P(z)$ with order m, then $A(z)$ has a multiplicative inverse $B(z)$, $P(z)$ has a multiplicative inverse $Q(z)$, and furthermore, $B(z)$ is an algebraic fractal having $Q(z)$ as a generator of order m.*
3. *If $P(z)$ is a generator of order m of the algebraic fractal $A(z)$, then $A(z) = A(z^m)P(z)$.*

Proof. 1. We can see that the statement is equivalent to saying $[A(z)]_m = [P(z)]_m$. Clearly, $\left[P\left(z^{m^i}\right)\right]_m = [1]_m$ if $i > 0$. So, $[A(z)]_m = \left(\prod_{r=0}^{\infty} \left[P\left(z^{m^r}\right)\right]_m\right) = [P(z)]_m$, as we wanted to see.

2. As we have seen in the previous point, since $m \geq 2$, $a_0 = p_0 = 1$. So, by Proposition 5.6, both $A(z)$ and $P(z)$ have multiplicative inverses, let them be denoted as $B(z)$ and $Q(z)$, respectively. Let $R(z) = P(z)Q(z) \equiv 1$. Clearly, $\prod_{r=0}^{\infty} R\left(z^{m^r}\right) = 1$. Therefore:

$$1 = \prod_{r=0}^{\infty} R\left(z^{m^r}\right) = \prod_{r=0}^{\infty} P\left(z^{m^r}\right) Q\left(z^{m^r}\right) = \prod_{r=0}^{\infty} P\left(z^{m^r}\right) \prod_{r=0}^{\infty} Q\left(z^{m^r}\right) = A(z) \left(\prod_{r=0}^{\infty} Q\left(z^{m^r}\right)\right)$$

As the second factor is a multiplicative inverse of $A(z)$, which is unique, it is equal to $B(z)$. Therefore, we can see that $Q(z)$ is a generator of $B(z)$ of order m.

3.

$$P(z)A(z^m) = P(z)\prod_{r=0}^{\infty} P\left((z^m)^{m^r}\right) = P(z)\prod_{r=0}^{\infty} P\left(z^{m^{r+1}}\right) = P(z)\prod_{s=1}^{\infty} P\left(z^{m^s}\right) = \prod_{s=0}^{\infty} P\left(z^{m^s}\right) = A(z).$$

□

At this point, it would be reasonable to formulate ourselves the following questions: For any $P(z) \in \mathbb{C}[[z]]$, and any integer $m \geq 2$, does there exist an algebraic fractal $A(z)$ having $P(z)$ as an algebraic fractal of order m? Similarly, for any $A(z) \in \mathbb{C}[[z]]$ with $[z^0]A(z) = 1$, and any integer $m \geq 2$, is $A(z)$ an algebraic fractal with some generator $P(z)$ of order m? And how about their uniqueness? Let us answer these questions:

Lemma 5.8. *Let $P(z) = \sum_{k=0}^{\infty} p_k z^n \in \mathbb{C}[[z]]$, with $p_0 = 1$, and let $m \geq 2$ be an integer. Then, exists a unique algebraic fractal $A(z) = \sum_{k=0}^{\infty} a_k z^n \in \mathbb{C}[[z]]$ having $P(z)$ as a generator of order m.*

Proof. We basically need to prove that the series $\prod_{r=0}^{\infty} P\left(z^{m^r}\right)$ converges to an element $A(z)$ of $\mathbb{C}[[z]]$, this will guarantee the existence and uniqueness of such $A(z)$. It will be enough with proving that $\forall n \in \mathbb{Z}_{>0}$ $\exists N_0 \in \mathbb{Z}_{>0}$ such that, $\forall N > N_0$, $\left[\prod_{r=0}^{N} P\left(z^{m^r}\right)\right]_n = \left[\prod_{r=0}^{N_0} P\left(z^{m^r}\right)\right]_n$.

If, for $n \in \mathbb{Z}_{>0}$, we take N_0 as the maximum integer such that $m^{N_0} < n$, we will have that $\forall k > N_0$, then $\left[P\left(z^{m^k}\right)\right]_n = [1]_n$. So, if $N > N_0$,

$$\left[\prod_{r=0}^{N} P\left(z^{m^r}\right)\right]_n = \prod_{r=0}^{N} \left[P\left(z^{m^r}\right)\right]_n = \prod_{r=0}^{N_0} \left[P\left(z^{m^r}\right)\right]_n = \left[\prod_{r=0}^{N_0} P\left(z^{m^r}\right)\right]_n,$$

as we wanted to see. □

Lemma 5.9. *Given $P(z) = \sum_{k=0}^{\infty} p_k z^k \in \mathbb{C}[[z]]$, with $p_0 = 1$, and an integer m such that $m \geq 2$, the solution of $X(z) = P(z)X(z^m)$, for $X(z) \in \mathbb{C}[[z]]$, exists and is unique except for a multiplicative constant, and a solution with $[z^0]X(z) = 1$ exists and is unique.*

Proof. By the previous lemma and by 5.7, a solution exists, which will be the algebraic fractal $A(z)$ having $P(z)$ as a generator of order m. Assume $X(z) \neq A(z)$ is also a solution. As $A(z)$ has a multiplicative inverse, we would have:

$$\frac{X(z)}{A(z)} = \frac{P(z)X(z^m)}{P(z)A(z^m)} = \frac{X(z^m)}{A(z^m)}$$

Setting $R(z) = \frac{X(z)}{A(z)}$, we would have $R(z) = R(z^m)$. If $R(z)$ is not a constant, we can write it as $R(z) = r_0 + z^n Q(z)$, for some $n \in \mathbb{Z}_{>0}$ and $Q(z) = \sum_{k=0}^{\infty} q_k z^k \in \mathbb{C}[[z]]$, with $q_0 \neq 0$. However, we would have $[z^n]R(z) = [z^n](r_0 + z^n Q(z)) = [z^n](r_0) + [z^n](z^n Q(z)) = [z^0]Q(z) = q_0 \neq 0$, whereas $[z^n](R(z^m)) = [z^n](r_0 + z^{nm} Q(z^m)) = [z^n](r_0) + [z^n](z^{nm} Q(z^m)) = 0$, getting a contradiction.

So, $\frac{X(z)}{A(z)}$ is a constant. Finally, as we have seen, a solution with $[z^0]X(z) = 1$ exists by setting $X(z) = A(z)$, and, as any other solution will be equal to $A(z)$ except for a multiplicative constant, $A(z)$ is the unique solution that satisfies $[z^0]X(z) = 1$, completing the proof. □

Another important lemma, is the following, regarding the uniqueness of generators of a given order.

Lemma 5.10. *Let $A(z) \in \mathbb{C}[[z]]$, with $[z^0]A(z) = 1$ and let $m \geq 2$ be an integer. If $A(z)$ is an algebraic fractal with some generator $P(z) \in \mathbb{C}[[z]]$ of order m, then such generator is unique.*

Proof. Let us show this by contradiction. Assume $A(z) = \prod_{r=0}^{\infty} P\left(z^{m^r}\right) = \prod_{r=0}^{\infty} Q\left(z^{m^r}\right)$, for some $P(z), Q(z) \in \mathbb{C}[[z]]$ such that $P(z) \neq Q(z)$ and $[z^0]P(z) = [z^0]Q(z) = 1$. As, $\forall r \in \mathbb{Z}_{\geq 0}$ we have $[z^0]\left(Q\left(z^{m^r}\right)\right) = 1$, by 5.6, $Q(z^{m^r})$ has a multiplicative inverse, and so:

$$1 = \frac{\prod_{r=0}^{\infty} P\left(z^{m^r}\right)}{\prod_{r=0}^{\infty} Q\left(z^{m^r}\right)} = \prod_{r=0}^{\infty} \frac{P\left(z^{m^r}\right)}{Q\left(z^{m^r}\right)} = \prod_{r=0}^{\infty} R\left(z^{m^r}\right),$$

where $R(z) = \frac{P(z)}{Q(z)}$. We can define such $R(z)$, since, by 5.6, $Q(z)$ has a multiplicative inverse. So, $R(z)$ is a generator of order m of the algebraic fractal $N(z) \equiv 1$. By 5.7, $N(z) = R(z)N(z^m)$, and so, $R(z) = 1$, meaning that $P(z) = Q(z)$ and therefore getting a contradiction. □

This leaves us with the following corollaries:

Corollary 5.11. *Let $A(z), P(z) \in \mathbb{C}[[z]]$, with $[z^0]A(z) = 1$, such that $A(z) = P(z)A(z^m)$. Then $P(z)$ is a generator of order m, and the unique generator of order m, of $A(z)$.*

Proof. Lemma 5.8 ensures us that exists $A(z)$ having $P(z)$ as a generator of order m, Proposition 5.7 ensures that the equality $A(z) = P(z)A(z^m)$ is satisfied, Lemma 5.9 ensures that, $A(z)$ is the only solution for $X(z) = P(z)X(z^m)$ such that $[z^0]A(z) = 1$, and Lemma 5.10 ensures $P(z)$ is the unique generator of order m of $A(z)$. □

Corollary 5.12. *Let $A(z) \in \mathbb{C}[[z]]$, with $[z^0]A(z) = 1$. Then, for any integer $m \geq 2$, we have that $A(z)$ is an algebraic fractal that has a unique generator $P(z)$ of order m, which can be obtained by setting $P(z) = \frac{A(z)}{A(z^m)}$.*

Proof. As $[z^0]A(z^m) = 1$, since the independent term is not affected if we transform z to z^m, $A(z^m)$ has a multiplicative inverse by Proposition 5.6, and so $P(z) = \frac{A(z)}{A(z^m)}$ can be defined. Corollary 5.11 ensures that $P(z)$ is a generator of $A(z)$ of order m, and the unique generator of $A(z)$ of order m. $\square$

5.2 Self-similar algebraic fractals

As we have seen so far, each formal power series $A(z)$ with complex coefficients that have an independent term equal to 1 can be seen as an algebraic fractal generated by some other power series $P(z)$ with complex coefficients and independent term equal to 1 for each possible order $m \geq 2$.

Therefore, we would want to study a more useful and specific case of algebraic fractals. As we will see, we will get a very interesting specific case when the algebraic fractal has a generator that is polynomial of degree strictly lower than its order. So, let us define the notion self-similar algebraic fractals, which will be a central concept in this work:

Definition 5.13. We will say that an algebraic fractal $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$ generated by $P(z) = \sum_{k=0}^{\infty} p_k z^k \in \mathbb{C}[[z]]$ with order $m \geq 2$ is self-similar if $P(z)$ satisfies that $p_0 = 1$, $p_i \neq 0$ for some i such that $0 < i < m$ and $p_i = 0$ if $i \geq m$.

Self-similar algebraic fractals have some special properties that make them different from the rest of algebraic fractals:

Proposition 5.14. *Again, with the same notation, self-similar algebraic fractals satisfy the following:*

1. *The degree of each monomial of the expansion of $\prod_{r=0}^{\infty} P\left(z^{m^r}\right)$ is unique.*
2. $a_k = p_k, \forall k \in \{0, \dots, m-1\}$
3. *Let s be a non-negative integer such that, expressed in base m, $s = (d_t d_{t-1} \cdots d_1 d_0)_m$. Then, $a_s = p_{d_t} \dots p_{d_0} = \prod_{i=0}^{t} p_{d_i}$.*
4. *Let s be a non-negative integer such that its expression in base m has c_1 1's, ..., c_{m-1} $(m-1)$'s. Then, $a_s = p_1^{c_1} \dots p_{m-1}^{c_{m-1}} = \prod_{i=1}^{m-1} p_i^{c_i}$.*
5. *$\forall b \in \{0, \dots m-1\}$ and $\forall r \in \mathbb{Z}_{\geq 0}$, we have $a_{bm^r} = p_b$.*
6. *$\forall b \in \{0, \dots m-1\}$ and $\forall r \in \mathbb{Z}_{\geq 0}$, we have $a_{b\frac{m^r-1}{m-1}} = p_b^r$.*

Proof. 1. We have that

$$\prod_{r=0}^{\infty} P\left(z^{m^r}\right) = (p_0 + p_1 z + \dots p_{m-1} z^{m-1})(p_0 + p_1 z^m + \dots p_{m-1} z^{(m-1)m}) \cdots$$

$$(p_0 z^{0m^0} + p_1 z^{1m^0} + \dots + p_{m-1} z^{(m-1)m^0})(p_0 z^{0m^1} + p_1 z^{1m^1} + \dots + p_{m-1} z^{(m-1)m^1}) \cdots$$

Therefore, any monomial of the expansion of this infinite product can be expressed as

$$(p_{i_0} p_{i_1} p_{i_2} \cdots p_{i_s}) z^{i_0 + i_1 m + i_2 m^2 + \dots + i_s m^s},$$

for some non-negative (and finite) value s, and with $i_s \neq 0$. In this case, the degree of the monomial, for such a choice of $s, i_0, \dots, i_s$, with $i_s \neq 0$, is a number that can be expressed as $(i_s i_{s-1} \dots i_1 i_0)_m$ in base m. As such choosing determines uniquely this m-base expression, which, at its turn, defines uniquely the value of the exponent, the degree of each monomial is unique.

2. True by Proposition 5.7.

3. It is an immediate consequence of the first point. As we have seen, there is a unique monomial with degree s, and its coefficient is $a_s = p_{d_0} p_{d_1} \cdots p_{d_t}$.

4. It is an immediate consequence of the previous point. If we take in account that $p_0 = 1$, we will have

$$a_s = \prod_{i=0}^{m-1} p_{d_i} = \left(\prod_{\substack{0 \le i < m \\ p_{d_i} = p_0}} p_0 \right) \left(\prod_{\substack{0 \le i < m \\ p_{d_i} = p_1}} p_1 \right) \cdots \left(\prod_{\substack{0 \le i < m \\ p_{d_i} = p_{m-1}}} p_{m-1} \right) =$$

$$\left(\prod_{\substack{0 \le i < m \\ p_{d_i} = p_1}} p_1 \right) \cdots \left(\prod_{\substack{0 \le i < m \\ p_{d_i} = p_{m-1}}} p_{m-1} \right) = p_1^{c_1} \cdots p_{m-1}^{c_{m-1}} = \prod_{i=1}^{m-1} p_i^{c_i}.$$

5. The m-base expression for bm^r has exactly one b, which is located in the r-th position, and zeroes elsewhere. Applying the fourth point of this Proposition, we have $a_{bm^r} = p_b^r$.

6. In a similar fashion, $b\frac{m^r-1}{m-1} = b(1 + m + \dots + m^{r-1})$. So its m-base representation has r b's (and nothing else). Again, applying the fourth point of this Proposition, we have $a_{b\frac{m^r-1}{m-1}} = p_b^r$.

□

We are interested in knowing what is the sum of the first coefficients of an algebraic fractal. So, let us define the concept of fractal sums:

Definition 5.15. We will define the fractal sum of the algebraic fractal $A(z)$ as the power series $S(z) = \sum_{k=0}^{\infty} s_k x^k$, with $s_k = \sum_{i=0}^{k} a_i$.

Observation 5.16. By the properties of generating functions, $S(z) = \frac{A(z)}{1-z}$.

We can show a couple of properties that these fractal sums have:

Proposition 5.17. *1. If $S(z) = \sum_{k=0}^{\infty} s_k z^k$ is the fractal sum of an algebraic fractal $A(z) \in \mathbb{C}[[z]]$, then $s_{k-1} = A_{[k]}(1)$.*

2. If $A(z)$ is an algebraic fractal having a generator $P(z)$ of order m, then its fractal sum $S(z)$ is an algebraic fractal with a generator $P(z)(1 + z + \dots + z^{m-1})$ of order m.

Proof. 1. $A_{[k]}(1) = \sum_{r=0}^{k-1} a_r = s_{k-1}$.

2. Applying Proposition 5.7,

$$S(z^m) = \frac{A(z^m)}{1 - z^m} = \frac{A(z)}{(1 - z^m)P(z)} = \frac{S(z)(1 - z)}{(1 - z^m)P(z)} = \frac{S(z)}{(1 + z + \dots + z^{m-1})P(z)}.$$

Applying Corollary 5.11, we complete the proof.

□

Let us state and prove the following two lemmas:

Lemma 5.18. *Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ be a self-similar algebraic fractal having a generator $P(z) = \sum_{k=0}^{m-1} p_k z^k$ of order m. Then, for every integer $s > 0$ and c such that $1 \leq c < m$, we have $A_{[cm^s]}(z) = \sum_{k=0}^{cm^s-1} a_k z^k = (P_{[c]}(z^s)) \prod_{r=0}^{s-1} P\left(z^{m^r}\right)$.*

Proof. First of all, let us note that saying $A_{[cm^s]}(z) = (P_{[c]}(z^s)) \prod_{r=0}^{s-1} P\left(z^{m^r}\right)$ is equivalent to saying $[A(z)]_{cm^s} = \left[\left(\sum_{k=0}^{c-1} p_k z^{km^t}\right) \prod_{r=0}^{s-1} P\left(x^{m^r}\right)\right]_{cm^s}$, since the polynomial of the RHS has degree smaller than cm^s. We will use the equivalence relation $\sim_{cm^s}$. For $t = s$, we will have:

$$\left[P(z^{m^t})\right]_{cm^s} = \left[\sum_{k=0}^{m-1} p_k z^{km^t}\right]_{cm^s} = \sum_{k=0}^{m-1} p_k \left[z^{km^t}\right]_{cm^s} = \sum_{k=0}^{c-1} p_k \left[z^{km^t}\right]_{cm^s} = \left[\sum_{k=0}^{c-1} p_k z^{km^t}\right]_{cm^s},$$

and, for $t \geq s$, $\left[P\left(z^{n^t}\right)\right]_{cm^s} = [1]_{cm^s}$. Therefore:

$$[A(z)]_{cm^s} = \left[\prod_{r=0}^{\infty} P\left(x^{m^r}\right)\right]_{cm^s} = \prod_{r=0}^{\infty} \left[P\left(x^{m^r}\right)\right]_{cm^s} =$$

$$\prod_{r=0}^{s-1} \left[P\left(x^{m^r}\right)\right]_{cm^s} \left[\sum_{k=0}^{c-1} p_k z^{km^t}\right]_{cm^s} = \left[\left(\sum_{k=0}^{c-1} p_k z^{km^t}\right) \prod_{r=0}^{s-1} P\left(x^{m^r}\right)\right]_{cm^s},$$

completing the proof. □

Lemma 5.19. *If $S(z) = \sum_{k=0}^{\infty} s_k z^k$ is the fractal sum of a self-similar algebraic fractal $A(z)$ that has a generator $P(z) = \sum_{k=0}^{m-1} p_k z^n$ of order m, then, for every non-negative integer r, $s_{m^r-1} = P(1)^r$.*

Proof. For $r = 0$ it is true since, as $S(z)$ is an algebraic fractal generated by some polynomial with order m by 5.17, we have $s_0 = 1$ by 5.7, and so, $s_{m^0-1} = s_0 = 1 = P^0(1)$. For $r > 0$, it is an immediate consequence of Lemma 5.18, just by setting $c = 1$, $z = 1$ (using the notation used in the Lemma). □

With these results, we can prove an important theorem, which will give us a recursive way of generating the coefficients of self-similar algebraic fractals.

Theorem 5.20. *Let $S(z) = \sum_{k=0}^{\infty} s_k z^k \in \mathbb{C}[[z]]$ be the fractal sum of the self-similar algebraic fractal $A(z) \in \mathbb{C}[[z]]$ generated by $P(z) = \sum_{k=0}^{m-1} p_k z^k \in \mathbb{C}[[z]]$ with order $m \geq 2$. Now, let $n \in \mathbb{Z}_{>0}$, and let $c \in \{1, \dots, m-1\}$ and $r \in \mathbb{Z}_{\geq 0}$ be the only two integers for which $cm^r \leq n < (c+1)m^r$ holds. Then, defining $s_{-1} = 0$, we have $s_{n-1} = s_{cm^r-1} + p_c s_{n-cm^r-1} = (p_0 + p_1 + \dots + p_{c-1}) s_{m^r-1} + p_c s_{n-cm^r-1} = (p_0 + p_1 + \dots + p_{c-1}) P^r(1) + p_c s_{n-cm^r-1}$.*

Proof. We have $s_{n-1} = \sum_{k=0}^{n-1} a_k = \sum_{k=0}^{cm^r-1} a_k + \sum_{k=cm^r}^{n-1} a_k$. By Lemma 5.18, $\sum_{k=0}^{cn^r-1} a_k = A_{[cm^r]}(1) = (p_0 + p_1 + \dots + p_{c-1}) P^r(1)$. By Lemma 5.19, we can see that $P^r(1) = s_{m^r-1}$, and by Proposition 5.17, $\sum_{k=0}^{cm^r-1} a_k = s_{cm^r-1}$.

We are left to see that $\sum_{k=cm^r}^{n-1} a_k = p_c s_{n-cm^r-1}$. In the special case $n = cm^r$, the equality trivially holds. Otherwise, as $\forall k \in \mathbb{Z}$ such that $k \in [cm^r, n-1]$, then $k = (d_r \cdots d_0)_m$, with $d_r = c$, we can see that $a_k = p_{d_r}(p_{d_{r-1}} \cdots p_{d_1} p_{d_0}) = p_c a_{(d_{r-1} \cdots d_0)_m} = p_c a_{k-cm^r}$. So, $\sum_{k=cm^r}^{n-1} a_k = \sum_{k=cm^r}^{n-1} p_c a_{k-cm^r} = p_c \sum_{j=0}^{n-cm^r-1} a_j = p_c s_{n-cm^r-1}$, as we wanted to see. □

An interesting corollary that can be deduced from the previous theorem, is the following:

Corollary 5.21. *Again, with the same conditions, if $n = (d_r \cdots d_0)_m$, defining $q_s = p_0 + \ldots + p_{s-1}$,*

$$s_{n-1} = \sum_{t=0}^{r} q_m^t q_{d_t} \prod_{i=t+1}^{r} p_{d_i}$$

Proof. Let us prove this by induction on r. If $r = 0$, then $d_0 = n < m$, and so, $s_{n-1} = a_0 + \ldots + a_{n-1} = p_0 + \ldots + p_{n-1} = q_n = q_{d_0}$. The second equality is true because of Proposition 5.7. So the equality of the statement is true for $r = 0$. Assume it is proven for $r - 1$. Now, if $n = (d_r d_{r-1} \cdots d_0)_m$, we can use Theorem 5.20 to see that $s_{n-1} = (p_0 + p_1 + \ldots + p_{c-1})s_{m^r-1} + p_c s_{n-cm^r-1}$, where $c = d_r$. As $n - cm^r = (d_{r-1} \cdots d_0)_m$, taking in account that, by Lemma 5.18, $s_{m^r-1} = P^r(1) = q_m^r$, we have

$$s_{n-1} = (p_0 + p_1 + \ldots + p_{c-1})s_{m^r-1} + p_c s_{n-cm^r-1} = q_{d_r} q_m^r + p_{d_r} \sum_{t=0}^{r-1} q_m^t q_{d_t} \prod_{i=t+1}^{r-1} p_{d_i} =$$

$$q_{d_r} q_m^r + \sum_{t=0}^{r-1} q_m^t q_{d_t} \prod_{i=t+1}^{r} p_{d_i} = \sum_{t=0}^{r} q_m^t q_{d_t} \prod_{i=t+1}^{r} p_{d_i}$$

Completing the proof. □

As we are treating with power series, it would be interesting to know about the analytical properties of $A(z)$. Before that, it would be necessary to introduce the big O notation.

Definition 5.22. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. We will define $\mathcal{O}(f)$ as $\mathcal{O}(g) = \{f : \mathbb{Z}_{\geq 0} \to \mathbb{R} | \exists c > 0, \exists n_0 \in \mathbb{Z}_{>0}$ such that $, \forall n > n_0, |f(n)| \leq c|g(n)|\}$

Definition 5.23. As it is usual, we will say that $\mathcal{O}(f) + \mathcal{O}(g) = \mathcal{O}(|f| + |g|)$ and $\mathcal{O}(f) \cdot \mathcal{O}(g) = \mathcal{O}(fg)$.

Notation. As usual as well, we will say that $f = \mathcal{O}(g)$ if and only if $f \in \mathcal{O}(g)$.

Proposition 5.24. *Let $f_1, f_2, g_1, g_2 : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(|g_1| + |g_2|)$ and $f_1 f_2 \in \mathcal{O}(g_1 g_2)$.*

We are ready to prove the following theorem:

Theorem 5.25. *Let $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$ be an algebraic fractal generated with order m by $P(z) = \sum_{k=0}^{\infty} p_k z^k \in \mathbb{C}[[z]]$. Assume $P(z)$ converges $\forall z \in \mathbb{C}$ such that $|z| < R$, for some $R \in \mathbb{R}_{>0}$. Then:*

1. *If $R < 1$, $A(z)$ converges $\forall z \in \mathbb{C}$ such that $|z| < R^{\frac{m}{m-1}}$*
2. *If $R \geq 1$, $A(z)$ converges $\forall z \in \mathbb{C}$ such that $|z| < 1$*
3. *The radius of convergence of a self-similar algebraic fractal is $R = 1$.*

Proof. 1. Let us remember that the radius of convergence R of a power series $B(z) = \sum_{k=0}^{\infty} b_k x^k$ is defined as $R = \frac{1}{\limsup_n \sqrt[n]{|b_n|}}$. Therefore, if $P(z)$ has radius of convergence R, this will mean that, $\forall \epsilon > 0$, $|p_n| = \mathcal{O}((\frac{1}{R} + \epsilon)^n)$. Reciprocally, it is immediate to see that, if $|p_n| = \mathcal{O}((\frac{1}{R})^n)$, then $P(z)$ converges $\forall z \in \mathbb{C}$ with $|z| < R$.

As $p_0 = 1$, we can express $A_{[n]}(z)$ as $A_{[n]}(z) = \prod_{r=0}^{\infty}(P_r)_{[n]}(z) = \prod_{r=0}^{s}(P_r)_{[n]}(z)$, where $P_i(z) = P\left(z^{m^i}\right)$ and s is the maximum integer for which $m^s < n$. So, we can bound $|a_n|$ by the sum of the absolute values of all the coefficients of the monomials of the expansion that will have degree smaller or equal than n:

$$|a_n| \leq \sum_{0 \leq i_0 + 2i_1 + \ldots + 2^s i_s < n} \leq |p_{i_0} p_{i_1} p_{i_2} \cdots p_{i_s}| \leq \sum_{\substack{0 \leq i_0 \leq n \\ 0 \leq i_1 \leq n/m \\ 0 \leq i_2 \leq n/m^2 \\ \vdots \\ 0 \leq i_s \leq n/m^s}} |p_{i_0} p_{i_1} p_{i_2} \cdots p_{i_s}| =$$

$$\left(\sum_{i=0}^{n} |p_i|\right) \left(\sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} |p_i|\right) \cdots \left(\sum_{i=0}^{\lfloor \frac{n}{m^s} \rfloor} |p_i|\right).$$

Now, we can see that, if $R < 1$, letting $M = \frac{1}{R} + \epsilon > 1$, for each integer t, $\sum_{i=0}^{t} |p_i| = \mathcal{O}((\frac{1}{R} + \epsilon)^0) + \ldots + \mathcal{O}((\frac{1}{R} + \epsilon)^t) = \mathcal{O}(M^0 + \ldots + M^t) = \mathcal{O}(\frac{M^{t+1}-1}{M-1}) = \mathcal{O}(M^t)$.

Therefore,

$$|a_n| \leq \left(\sum_{i=0}^{n} |p_i|\right) \left(\sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} |p_i|\right) \cdots \left(\sum_{i=0}^{\lfloor \frac{n}{m^s} \rfloor} |p_i|\right) =$$

$$\mathcal{O}(M^n)\mathcal{O}(M^{\frac{n}{m}}) \cdots \mathcal{O}(M^{\frac{n}{m^s}}) = \mathcal{O}\left(M^{n(1+\frac{1}{m}+\ldots+\frac{1}{n^s})}\right) = \mathcal{O}\left(M^{n\frac{m}{m-1}}\right)$$

So, as $|a_n| = \mathcal{O}\left(M^{n\frac{m}{m-1}}\right)$, the radius of convergence of $A(z)$ will be bigger or equal than $\frac{1}{M^{\frac{m}{m-1}}} = \frac{1}{(\frac{1}{R}+\epsilon)^{\frac{m}{m-1}}}$. Taking $\epsilon \to 0$, $A(z)$ will converge $\forall z \in \mathbb{C}$ with $|z| < R^{\frac{m}{m-1}}$.

2. Now, to prove this, if $P(z)$ converges $\forall z \in \mathbb{C}$ with $|z| < R$, for $R \geq 1$, it will also converge for any $R' < 1$. So, it will also converge $\forall z \in \mathbb{C}$ with $|z| < R'^{\frac{m}{m-1}}$. Taking $R' \to 1^-$, we get the desired result.

3. Finally, let us prove the last point. If $P(z)$ is the generator of order m of the self-similar algebraic fractal, then, by definition $P(z) = 1 + p_1 z + \ldots + p_{m-1} z^{m-1}$, with $p_c \neq 0$, for some $c \in \{1, \ldots m-1\}$. As $P(z)$ is a polynomial, it has infinite radius of convergence, so $A(z)$ converges $\forall z \in \mathbb{C}$, with $|z| < 1$, as a consequence of the previous point we just proved, meaning that its radius of convergence R satisfies $R \geq 1$.

   However, $a_{cm^s} = p_c$, $\forall s \in \mathbb{Z}_{\geq 0}$ as we have seen in 5.7. As the sequence $(c, cm, cm^2, cm^3, \ldots)$ is increasing and unbounded, we will have $1 \leq R = \frac{1}{\limsup_n \sqrt[n]{|a_n|}} \leq \frac{1}{\limsup_s \sqrt[cm^s]{p_c}} = 1$, meaning $R = 1$, and completing the proof.

□

Finally, we will show an interesting connection between self-similar algebraic fractals and tensors:

Theorem 5.26. *Let $A(z)$ be a self-similar algebraic fractal having a generator $P(z) = \sum_{k=0}^{m-1} p_k x^k$ of order m. Let E be an m-dimensional vector space over the field of complex numbers, and let $\mathcal{B} = \{e_0, \ldots, e_{m-1}\}$*

be a basis of this space. Let $T_P : E \to \mathbb{C}$ be the linear map (or 1-covariant tensor) defined by $T_P = p_0 e_0^* + \dots + p_{n-1} e_{m-1}^*$. For any integer r, let us denote by T_P^r the r-fold tensor product of T_P, namely, $T_P^r = \underbrace{T_P \otimes \cdots \otimes T_P}_{n \text{ times}}$.

Then, we have that, for each $s \in \mathbb{Z}_{>0}$, T_P^s is a symmetric s-covariant tensor that can be written as

$$T_P^s = \sum_{0 \le i_0, \dots, i_{s-1} < m} t_{i_0, i_1, \dots i_{s-1}} e_{i_0}^* \otimes e_{i_1}^* \otimes \cdots \otimes e_{i_{s-1}}^*,$$

and such that, for all $n \in \mathbb{Z}_{\ge 0}$ with $n < m^s$, if the m-base expression for n is $n = (d_{s-1} d_{s-2} \cdots d_1 d_0)_n$ (probably with some zeros on the left, or all made up of zeroes if $m = 0$), then $t_{d_0, d_1, \dots, d_{s-1}} = a_n$.

Proof. We will show this by induction. The statement trivially holds if $s = 1$. Now, let us assume that the statement is true for s. We have

$$T_P^{s+1} = T_P^s \otimes T_P = \left(\sum_{0 \le i_0, \dots, i_{s-1} < m} t_{i_0, i_1, \dots i_{s-1}} e_{i_0}^* \otimes e_{i_1}^* \otimes \cdots \otimes e_{i_{s-1}}^* \right) \otimes \left(\sum_{k=0}^{m-1} p_k e_k^* \right) =$$

$$\sum_{0 \le i_0, \dots, i_{s-1}, k < m} t_{i_0, i_1, \dots i_{s-1}} p_k e_{i_0}^* \otimes e_{i_1}^* \otimes \cdots \otimes e_{i_{s-1}}^* \otimes e_k^*.$$

We therefore have now that T_P^s is an s-covariant tensor, and that $t_{d_0, d_1, \dots d_{s-1} d_s} = t_{d_0, d_1, \dots d_{s-1}} p_{d_s} = a_n p_{d_s}$, with $n = (d_0 d_1, \dots d_{s-1})_n < n^s$. As we have seen before in proposition 5.14, $a_n = p_{d_0} p_{d_1} \cdots p_{d_{s-1}}$, and so, $a_n p_{d_s} = p_{d_0} p_{d_1} \cdots p_{d_{s-1}} p_{d_s} = a_r$, where $r = n + d_s m^s$ is the positive integer whose m-base representation is $r = (d_s d_{s-1} \dots d_1 d_0)_m$, proving the induction.

Now, we can see that, $\forall s \ge 0$, as $t_{i_0, i_1, \dots i_{s-1}} = p_{i_0} p_{i_1} \cdots p_{i_{s-1}}$, its value does not depend on the order of the indices, proving the symmetry of T_P^s. □

5.3 Integer fractals and geometric interpretation

We have developed some theory for algebraic fractals with special emphasis in the case of the self-similar ones. Now, let us focus on the case in which all the coefficients of the algebraic fractal are elements of a certain subset of the complex numbers that satisfy some property. Let us start with the following definition:

Definition 5.27. Let $\mathbb{X} \subset \mathbb{C}$. We will define the set $\mathbb{X}[[z]]$ as $\mathbb{X}[[z]] = \{B(z) = \sum_{k=0}^{\infty} b_k z^k : b_k \in \mathbb{X}\ \forall k \in \mathbb{Z}_{\ge 0}\}$.

Observation 5.28. Clearly, if $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the complex numbers such that $\mathbb{X} \subset \mathbb{Y}$, then $\mathbb{X}[[z]] \subset \mathbb{Y}[[z]]$. In particular, if $\mathbb{Y} = \mathbb{C}$, we have $\mathbb{X}[[z]] \subset \mathbb{C}[[z]]$.

We can give some interesting properties about self-similar fractals the coefficients of which are in some particular subset of the complex numbers:

Proposition 5.29. *Let $\mathbb{X} \subset \mathbb{C}$ be a set such that $\mathbb{X}$ is closed under multiplication and such that $1 \in \mathbb{X}$. Let $A(z)$ be a self-similar algebraic fractal and $P(z)$ be a generator of order m. Then, $A(z) \in \mathbb{X}[[z]]$ if and only if $P(z) \in \mathbb{X}[[z]]$.*

Proof. As we have seen in proposition 5.7, $a_k = p_k, \forall k \in \{0, \dots m-1\}$. So if $P(z) \notin \mathbb{X}[[X]]$, then $A(z) \notin \mathbb{X}[[z]]$. We can therefore assume $P(z) \in \mathbb{X}[[z]]$. However, as we have seen in 5.14, if s is a positive integer such that its m-base expression is $s = (d_r \dots d_0)_m$, we will have $a_s = p_{d_0} \cdots p_{d_r}$. As the product $p_{d_0} \cdots p_{d_r}$ is a product of elements of $\mathbb{X}$, which is closed under $\mathbb{X}$, we have $a_s \in \mathbb{X}$, as we wanted to show. □

Proposition 5.30. *Let $\mathbb{X} \subset \mathbb{C}$ be a set such that $\mathbb{X}$ is closed under addition and such that $1 \in \mathbb{X}$. Let $A(z)$ be a self-similar algebraic fractal and $P(z)$ be a generator of order m. Let $S(z)$ be the fractal sum of $A(z)$. Then if $A(z) \in \mathbb{X}[[z]]$, then $S(z) \in \mathbb{X}[[z]]$ as well. The reciprocal is true if $\mathbb{X}$ is closed under subtraction.*

Proof. If $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{X}[[z]]$, as $S(z) = \sum_{k=0}^{\infty} s_k z^k$, with $s_k = \sum_{i=0}^{k} a_k$, we have $s_k \in \mathbb{X}$, since the sum in $\mathbb{X}$ is closed, and so, $S(z) \in \mathbb{X}[[z]]$.

The reciprocal is also true if $\mathbb{X}$ is closed under subtraction. Assume it is not, then $a_k \notin \mathbb{X}$ for some k. Let $r \geq 0$ be the minimum integer such that $a_r \notin \mathbb{X}$. If $r = 0$, we would have $a_0 = s_0 \in \mathbb{X}$, getting a contradiction. Otherwise, if $r > 0$ we would have $a_r = s_r - s_{r-1} \in \mathbb{X}$, since $\mathbb{X}$ is closed under subtraction, getting a contradiction. □

Corollary 5.31. *Let $\mathbb{X}$ be either $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Let $A(z)$ be a self-similar algebraic fractal and $P(z)$ be a generator of order m. Let $S(z)$ be the fractal sum of $A(z)$. Then, $A(z), P(z), S(z)$ have multiplicative inverses $B(z), Q(z), T(z)$, and for any $X(z), Y(z) \in \{A(z), B(z), P(z), Q(z), S(z), T(z)\}$, we have $X(z) \in \mathbb{X}[[z]]$ if and only if $Y(z) \in \mathbb{X}[[z]]$.*

Proof. The sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are closed under addition, subtraction, and multiplication. Now, as $[z^0]A(z) = [z^0]P(z) = [z^0]S(z) = 1$, Proposition 5.6 ensures that such inverses exist, and the construction of multiplicative inverses ensures that, if $F(z) \in \mathbb{C}[[z]]$, with $[z^0]F(z) = 1$, and $G(z) \in \mathbb{C}[[z]]$ is it multiplicative inverse, then $F(z) \in \mathbb{X}[[z]]$ if and only if $G(z) \in \mathbb{X}[[z]]$. To complete the proof of this corollary, it is enough to show that $A(z) \in \mathbb{X}[[z]]$ if and only if $P(z) \in \mathbb{X}[[z]]$ if and only if $S(z) \in \mathbb{X}[[z]]$, which is immediate from the two previous propositions. □

Before moving on, let us define a simplified form to write a self-similar algebraic fractal in terms of a polynomial generator:

Definition 5.32. Let $p_0, \dots, p_{m-1} \in \mathbb{C}$ be n complex numbers with $p_0 = 1$ and such that $p_i \neq 0$ for some i such that $0 < i < m$. We will denote by $[p_0, \dots, p_{m-1}]$ the self-similar algebraic fractal having the polynomial $P(z) = \sum_{k=0}^{m-1} p_k z^k$ as a generator of order m.

Having seen this, we can define the notion of integer fractals.

Definition 5.33. We will say that an algebraic fractal $A(z) \in \mathbb{C}[[z]]$ is an integer fractal if $A(z) \in \mathbb{Z}_{\geq 0}[[z]]$.

Once we have this definition, we can see, with some examples, why these algebraic fractals are useful in order to describe self-similar fractals:

Example 5.34. Let $P(z) = 1 + z^2$. Let $A(z)$ be the self-similar algebraic fractal of order $m = 3$ generated by $P(z)$ (in other words, $A(z) = [1, 0, 1]$). The terms with degree smaller or equal that $m^3 = 27$, will be given by $P(z)P(z^3)P(z^9)$, which, expanded, yields:

$$P(z)P(z^3)P(z^9) = z^0 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26}.$$

Figure 5: Cantor set after 3 iterations. Starting from a single interval, at each iteration the interval is partitioned in three equal intervals and the middle one is erased

Now, let us look at the Cantor set iterated 3 times, which we can see in Figure 5.

If each square has side length one, the whole figure has length $3^3 = 27$. If the leftmost position is the zeroth position, one can see that the marked positions are $0, 2, 6, 8, 18, 20, 24, 26$, which are the coefficients of the polynomial $P(z)P(z^3)P(z^9) = z^0 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26}$. As we will see this is not a coincidence.

Example 5.35. Let $P(z) = 1 + 2z$. Let $A(z)$ be the self-similar algebraic fractal of order $m = 2$ generated by $P(z)$ (in other words, $A(z) = [1, 2]$). The terms with degree smaller or equal that $m^4 = 16$, will be given by $P(z)P(z^2)P(z^4)P(z^8)$, which, expanded, yields:

$$P(z)P(z^2)P(z^4)P(z^8) =$$

$$z^0 + 2z^1 + 2z^2 + 4z^3 + 2z^4 + 4z^5 + 4z^6 + 8z^7 + 2z^8 + 4z^9 + 4z^{10} + 8z^{11} + 4z^{12} + 8z^{13} + 8z^{14} + 16z^{15}.$$

Now, let us look at the Sierpinski Triangle constructed after 4 iterations, which we can see in Figure 6.

Now, we have $2^4 = 16$ rows, and if we consider the row on the top to be zeroth row, we can see that the coefficient of degree k in the previous polynomial is equal to the number of small triangles on the k-th row.[6]

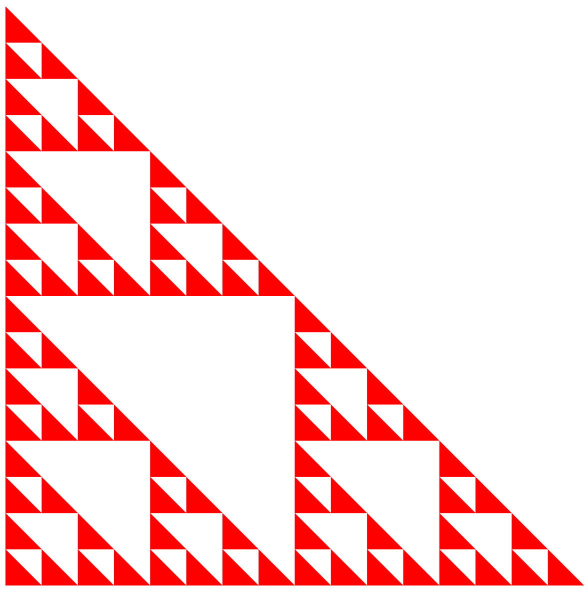

Figure 6: Sierpinski triangle constructed after 4 iterations

[6] As an interesting remark: It is well known that if one takes Pascal's triangle, and erases the numbers which are even, the shape they form is the Sierpinski's triangle. Similarly, if one does the same for some other prime number p, and erases the elements that are multiple of p, one can see that the shape they form is a fractal such that the number of elements in each row in generated by $[1, 2, \ldots, p]$.

Now we are ready to present the geometrical interpretation of integer fractals. Let $p_1, \dots, p_{m-1}$ be non-negative integers. Let us construct iteratively a fractal the following way. We will start with a block made up by just one element (in the case of the Sierpinski's triangle, for example, it is the small triangle on the top), and at each iteration, copy the current block p_1 times right below the current block (each of the p_1 copies at the same height), p_2 times right below those (each of the p_2 copies at the same height), and so on until copying it a total of p_{m-1} times at the bottom (again, each of the p_{m-1} copies at the same height). Now, let $P(z) = \sum_{k=0}^{m-1} p_k z^k \in \mathbb{Z}_{\geq 0}[[z]]$, with $p_0 = 1$. Then, the self-similar integer fractal $A(z) = \sum_{k=0}^{\infty} a_k z^k$, having $P(z)$ as a generator of order m, that is, $A(z) = [1, p_1, \dots, p_{m-1}]$, which, by Corollary 5.31, means $A(z) \in \mathbb{Z}_{\geq 0}[[z]]$, satisfies that, $\forall k \in \mathbb{Z}_{\geq 0}$, the coefficient a_k gives how many copies of the original element there are in the k-th row of the fractal, where the indexing starts by 0.

The concept of integer fractals will not be useful enough to develop the next section, since, as we will see, the restriction of having non-negative integer coefficient will be too strong for what we want. However, this geometric interpretation will be extremely useful from now on, as we will see.

5.4 Boolean fractals and geometric interpretation

Once we have introduced the notion of integer fractals and once we have discussed its geometric interpretation, a natural step would be to try to generalize them. So, we can try to study what happens if the coefficients are still integer values but not restricted to the condition of being non-negative.

Definition 5.36. We will say that an algebraic fractal $A(z) \in \mathbb{C}[[z]]$ is a boolean fractal if $A(z) \in \mathbb{Z}[[z]]$.

Let us see an example to understand them better.

Example 5.37. The Thue-Morse sequence is a sequence of bits defined the following way: We start with a string of just one zero. At each step, we append this string to the end of itself, but flipping the bits, namely, changing zeroes for ones and ones for zeroes. The sequence of strings we would have at each iteration, therefore, would be $0, 01, 0110, 01101001, 0110100110010110, \dots$. The sequence we get as the number of iterations goes to infinity is the Thue-Morse sequence.

We can see the Thue-Morse sequence as an algebraic fractal. Before that, we will define as the modified Thue-Morse sequence the sequence which is created by changing each 0 of the Thue-Morse sequence for a 1, and each 1 for a -1 (in other words, if b_n is the n-th bit of the Thue-Morse sequence, the n-th element of the modified version, a_n, satisfies $a_n = (-1)^{b_n}$). So, if $A(z) = \sum_{k=0}^{\infty} a_k z^k$ is the generating function of the modified Thue-Morse sequence, which means that a_k is the k-th bit of the sequence (starting from zero), then $A(z)$ is a self-similar fractal having the following generator of order 2: $P(z) = 1 - z$. In other words, $A(z) = [1, -1]$.

Figure 7: Thue-Morse sequence after $m = 4$ iterations. In red, the bits equal to 0 (or equal to 1 in the modified Thue-Morse sequence). In green, the ones equal to 1 (or to -1 in the modified version).

We can deduce from here an interesting property of this sequence. By Proposition 5.14, a_n will be $(1)^{z_n}(-1)^{o_n} = (-1)^{o_n}$, where z_n, o_n represent the number of zeroes and ones in the binary representation of n. So, a_n will be 1 if n has an even number of zeroes in its binary representation and -1 otherwise, and equivalently, the n-th bit of the Thue-Morse sequence will be 0 in the first case, and 1 in the other.

In Figure 7, we can see what the Thue-Morse looks like after four iterations.

Now, let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ be a self-similar boolean fractal that comes generated with order $m \geq 2$ by the polynomial $P(z) = \sum_{k=0}^{m-1} p_k z^k \in \mathbb{Z}[[z]]$. Again, by Corollary 5.31, we have $A(z) \in \mathbb{Z}[[z]]$. Let us write p_k as $p_k = q_k \cdot (-1)^{s_k}$, where $q_k = |p_k|$ and $s_k = 0$ if $p_k \geq 0$ or $s_k = 1$ otherwise. Now, if $n = (d_r \ldots d_0)_m$, we have $a_n = p_{d_r} \cdots p_{d_0} = q_{d_r} \cdots q_{d_0} (-1)^{s_{d_r} + \ldots + s_{d_0}} = q_{d_r} \cdots q_{d_0} (-1)^{(s_{d_r} + \ldots + s_{d_0}) \mod 2}$.

The identification $p_i = q_i \cdot (-1)^{s_i}$ allows us to rethink each coefficient of the self-similar boolean fractal as a pair consisting in a non-negative integer and a boolean value. Multiplying some coefficients $p_{d_r} \cdots p_{d_0}$ is equivalent to a pair in which the integer value is the product of the absolute values $q_{d_r} \cdots q_{d_0}$, and the boolean is the sum modulo 2 of $s_{d_r} + \ldots + s_{d_0}$. This sum can also be expressed as $s_{d_r} \oplus \ldots \oplus s_{d_0}$, where $\oplus$ is the *XOR* operator[7].

With this, we can give a geometrical interpretation to self-similar boolean fractals. If $P(z) = \sum_{k=0}^{m-1} p_k z^k$ is a generator of order $m \geq 2$ of such self-similar boolean fractal, we can think in a fractal that has $|p_0| = 1$ elements in its zeroth row, $|p_1|$ in the first, and so on, until having $|p_{m-1}|$ elements in the $(m-1)$-th, in the same sense as in integer fractals. However, each of these elements will have associated to them a boolean value. We can think of them as red for 0 (False) values and green for 1 (True) values. At each iteration, the fractal will evolve as we have seen with integer fractals, but, this time, the copies originated from a green element will have its colours inverted from the original.

In the following section, we will see how this geometrical interpretation will be extremely useful, since it will allow us to create an object which will be closely related to the Collatz problem.

[7] This is the reason why we have called them "boolean" fractals. The names "integer fractal" and "boolean fractal" are chosen from a computational perspective and not from a mathematical one

6. Reformulating the weak Collatz conjecture

The goal of this section is to present the main result of this work by combining the results of the previous sections.

In Section 4, we have introduced the weak Collatz conjecture and presented the equivalent statement of the conjecture due to Böhm and Sontacchi [6].

Let us give another look at its formulation (Conjecture 4.2). As we can see, the statement of the conjecture involves the following formula:

$$\left(2^b - 3^{k+1}\right) n = \sum_{j=0}^{k} 3^{k-j} 2^{a_j}$$

Something surprising can come to one's mind when seeing this formula. To that purpose, let us look back at the last result we obtained in Section 3, which was Theorem 3.6. This theorem stated that, if $n = 2^{a_k} + \ldots + 2^{a_0}$, the number of elements in the first n rows of Sierpinski's triangle is $s(n) = \sum_{j=0}^{k} 3^{a_j} 2^{k-j}$.

We may realize that the right hand side of the equality stated in the alternative statement of the weak Collatz conjecture, $\sum_{j=0}^{k} 3^{k-j} 2^{a_j}$, is practically identical to the formula of how many small triangles are there in the first n rows of the Sierpinski triangle we have seen earlier, with the only difference that the 3's and the 2's are switching roles.

In the case of the formula of Sierpinski's triangle, that formula was the result of the recurrence

$$s(n) = \begin{cases} 3^k & \text{if } n = 2^k \\ s(2^k) + 2s(n - 2^k) & \text{if } 2^k < n < 2^{k+1} \end{cases}.$$

So, it seems a good idea to generate a similar recurrence, but now switching the roles of the 2's and the 3's. Doing so, we get the following lemma:

Lemma 6.1. *Let $r(n)$ be the recurrence defined, only in the set of strictly positive integers that satisfy that its base 3 expression have no 2's, by:*

$$r(n) = \begin{cases} 2^k & \text{if } n = 3^k \\ r(3^k) + 3r(n - 3^k) & \text{if } 3^k < n < 2 \cdot 3^{k+1} \end{cases}.$$

Now, let $n \in \mathbb{Z}_{>0}$ such that, if we express n in base 3, all its digits are 0's and 1's, or equivalently, such that $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$ for some integers $k \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_k$ with $0 \leq a_0 < a_1 < \ldots < a_k$.

Then, $r(n) = \sum_{j=0}^{k} 3^{k-j} 2^{a_j}$.

Proof. This proof will be almost identical to the proof of Theorem 3.6. We will proceed by induction on k. If $k = 0$, $n = 3^{a_0}$ and so, $r(n) = r(3^{a_0}) = 2^{a_0}$, which satisfies the formula.

We will assume now that the statement has been proven for $k-1$. Now, if $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_0}$, we clearly have $3^{a_k} < n < 2 \cdot 3^{a_k}$, and so,

$$r(n) = r(3^{a_k}) + 3r(n - 3^{a_k}) = 2^{a_k} + 3r(3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_0} - 3^{a_k}) =$$

$$2^{a_k} + 3r(3^{a_{k-1}} + \ldots + 3^{a_0}) = 2^{a_k} + 3\sum_{j=0}^{k-1} 2^{a_j} 3^{k-1-j} = 2^{a_k} + \sum_{j=0}^{k-1} 2^{a_j} 3^{k-j} = \sum_{j=0}^{k} 2^{a_j} 3^{k-j}.$$

As we wanted to show. ☐

As we can see, we can generate, with a recursive formula similar to the one we used with Sierpinski's triangle, a formula similar to that we used to count the number of elements in the first n rows of Sierpinski's triangle. So, what we would want to do now, is to find a fractal such that the sum of elements of the first n rows satisfies the recursive formula of $r(n)$. However, we want to index this rows starting from 0. and therefore, we will define $t(n) = r(n+1)$. So, $t(n)$ will satisfy the following recursive formula:

$$t(n-1) = \begin{cases} 2^k & \text{if } n = 3^k \\ t(3^k - 1) + 3t(n-1-3^k) & \text{if } 3^k < n < 2 \cdot 3^{k+1} \end{cases},$$

and it will satisfy that, if $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$ for some integers $k \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_k$ with $0 \leq a_0 < a_1 < \ldots < a_k$, then $t(n-1) = \sum_{j=0}^{k} 2^{a_j} 3^{k-j}$.

Now, by looking at Theorem 5.20, we may see that the recurrence formula

$$t(n-1) = t(3^k - 1) + 3t(n - 3^k - 1)$$

is a particular case of the formula

$$t_{n-1} = t_{cm^k-1} + p_c t_{n-cm^k-1} = (p_0 + p_1 + \ldots + p_{c-1})P^k(1) + p_c t_{n-cm^k-1}$$

which is the one that determines the $(n-1)$-th coefficient, with $cm^k \leq n < (c+1)m^k$, of the fractal sum of some self-similar algebraic fractal generated by a polynomial $P(z) = \sum_{r=0}^{m-1} p_r z^r \in \mathbb{C}[[z]]$, with $p_0 = 1$. Let us try to adapt this recursive formula so it is as similar as possible to $t(n-1)$. In this particular case, we will have $m = 3$, meaning that $P(z)$ will have degree at most $m - 1 = 2$, and $c = 1$. For this case, we have $\sum_{r=0}^{c-1} p_r = p_0 = 1$, and so, we can simplify this recurrence to

$$t_{n-1} = t_{3^k-1} + p_1 t_{n-3^k-1} = P^k(1) + p_1 t_{n-3^k-1}.$$

If we want both recurrences to be equal, $\forall k \in \mathbb{Z}_{\geq 0}$ we will need $p_1 = 3$, and $P^k(1) = t_{3^k-1} = 2^k$, meaning $p_0 + p_1 + p_2 = P(1) = 2$, and so, as $p_0 = 1$, we have $p_2 = 2$. Therefore, we are left with $P(z) = 1 + 3z - 2z^2$.

With this in mind, we are now ready to prove the following theorem:

Theorem 6.2. *Let $P(z) = 1 + 3z - 2z^2$, and let $A(z)$ be the self-similar algebraic fractal that has $P(z)$ as a generator of order $m = 3$ (in other words, $A(z) = [1, 3, -2]$). Let $T(z) = \sum_{k=0}^{\infty} t_k z^k = \frac{A(z)}{1-z}$ be its fractal sum.*

Then, if $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$ for some integers $a_0, \ldots, a_k$ such that $a_k > a_{k-1} > \cdots > a_1 > a_0 \geq 0$, we have $t_{n-1} = \sum_{j=0}^{k} 3^{k-j} 2^{a_j}$.

Proof. It will be enough with seeing that, if $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$ for some integers $a_0, \ldots, a_k$ such that $a_k > a_{k-1} > \cdots > a_1 > a_0 \geq 0$, we have $t(n-1) = t_{n-1}$, or, in other words, it will be enough with proving that t_{n-1} has the same recursive formula as $t(n-1)$, and that, for the special cases in which $n = 3^k$, we will have $t(n-1) = t_{n-1} = 2^k$. Let us prove that the latter holds: $t_{3^k-1} = t(3^k - 1) = 2^k$, which will be true by Lemma 5.19. Finally, as we have seen, Theorem 5.20 will ensure that, as n satisfies $c \cdot 3^k \leq n < (c+1) \cdot 3^k$, with $c = 1$, we will have

$$t_{n-1} = t_{3^k-1} + p_1 t_{n-3^k-1} = t_{3^k-1} + 3t_{n-3^k-1}.$$

So, $t_{n-1} = t(n-1) = \sum_{j=0}^{k} 3^{k-j} 2^{a_j}$, completing the proof. [8] [9] ∎

[8] The sequence of values $t(n-1)$, with $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$, seems to be the sequence OEIS A119733.

[9] One can see that the simplicity of the formula $\sum_{j=0}^{k} 2^{a_j} 3^{k-j}$ is a consequence of Corollary 5.21.

From now on, $P(z)$ will always be the the polynomial $P(z) = 1 + 3z - 2z^2$, $A(z) = [1, 3, -2]$ will be the self-similar fractal generated by $P(z)$ with order 3, and $T(z)$, its fractal sum.

So, we have seen that the formula $\sum_{j=0}^{k} 2^{a_j} 3^{k-j}$ is the $(n-1)$-th coefficient of the fractal sum of $[1, 3, -2]$, with $n = 3^{a_k} + 3^{a_{k-1}} + \ldots + 3^{a_1} + 3^{a_0}$.

Before moving on, we can see some properties of $A(z)$ and $T(z)$ by reviewing some of the theory we have developed in the previous section:

Proposition 6.3. *1. $T(z)$ is an algebraic fractal having $Q(z) = 1 + 4z + 2z^2 + z^3 - 2z^4$ as a generator of order 3.*

2. $A(z) = (1 + 3z - 2z^2)A(z^3)$ and $T(z) = (1 + 4z + 2z^2 + z^3 - 2z^4)T(z^3)$.

3. The radius of convergence of $A(z)$ is exactly 1, and $T(z)$ converges $\forall z \in \mathbb{C}$ such that $|z| < 1$.

Proof. 1. Immediate from Proposition 5.17, since $(1+3z-2z^2)(1+z+z^2) = (1+4z+2z^2+z^3-2z^4)$.

2. Immediate from Proposition 5.7.

3. Immediate from Theorem 5.25.

□

Once we have seen this, we may realize that $P(z) \in \mathbb{Z}[[z]]$ and $P(z) \notin \mathbb{Z}_{\geq 0}[[z]]$, and so, by Corollary 5.31 and by Proposition 5.29, $A(z) \in \mathbb{Z}[[z]]$, $T(z) \in \mathbb{Z}[[z]]$, and $A(z) \notin \mathbb{Z}_{\geq 0}[[z]]$, meaning that $A(z)$ is not an integer fractal, like in the case of the Sierpinski Triangle, but a boolean one. Therefore we can use its geometrical interpretation that we have seen in Section 5.4. So, we can generate the pattern shown in Figure 8 (which we iterate in Figures 9 and 10), which we will name as Collatz boolean fractal[10].

Now, with all we have seen so far, we are ready to present an equivalent of the weak Collatz conjecture, which we will do by presenting two different equivalent statements:

Conjecture 6.4 (Reformulated weak Collatz conjecture, algebraic version)**.** *Let $T(z) = \sum_{k=0}^{\infty} t_k z^k$ be the algebraic fractal having as a generator of order 3 the polynomial $Q(z) = 1 + 4z + 2z^2 + z^3 - 2z^4$ (or equivalently, the fractal sum of $A(z) = [1, 3, -2]$).*

Let $k \in \mathbb{Z}_{\geq 0}$, and let $a_0, \ldots, a_k$ be integers such that $0 = a_0 < a_1 < \cdots < a_k$. Then, there does not exist $b > a_k$ such that $2^b - 3^{k+1}$ is a positive number that is a proper divisor of t_{n-1}, with $n = 3^{a_k} + \ldots + 3^{a_0}$.

Conjecture 6.5 (Reformulated weak Collatz conjecture, geometric version)**.** *Let $k \in \mathbb{Z}_{\geq 0}$, let $a_0, \ldots, a_k$ be integers such that $0 = a_0 < a_1 < \cdots < a_k$ and let $n = 3^{a_k} + \ldots + 3^{a_0}$. Let t_{n-1} be the number of red elements that are between the rows 0 and $n-1$ of the Collatz boolean fractal minus the number of green ones in the same range. Then, there does not exist $b > a_k$ such that $2^b - 3^{k+1}$ is a positive number that is a proper divisor of t_{n-1}.*

We can add a couple observations:

Observation 6.6. We can give a geometrical meaning to 2^b and 3^{k+1}. By Lemma 5.18, $2^b = P^b(1) = s_{3^b - 1}$, and, by Proposition 5.14, letting r be a number the base 3 expression of which consists of some 0's, no 2's and $k+1$ $1'$s, we have $3^{k+1} = a_r$. So, 2^b is the number of red elements minus the number of green elements between rows 0 and $3^b - 1$ of the Collatz boolean fractal, and 3^{k+1} is the number of (red) elements in row r.

[10] We have used this particular pattern for being the most compact and symmetrical.

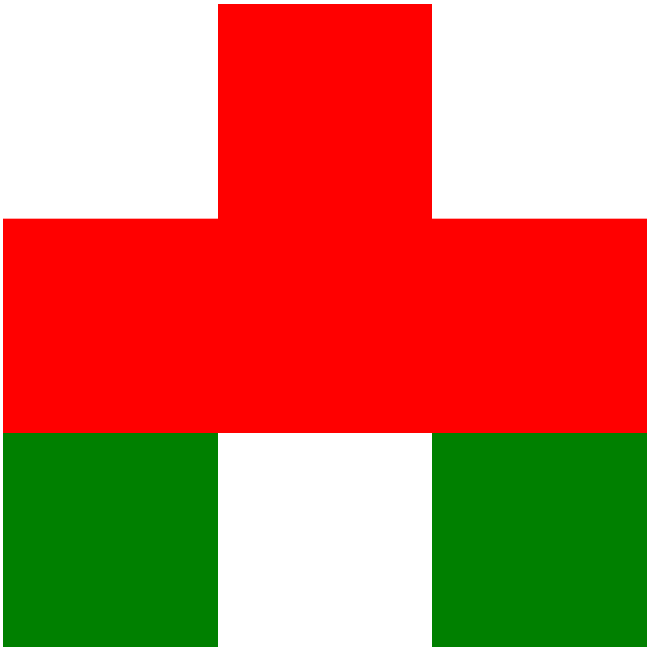

Figure 8: Pattern from where the Collatz Boolean fractal is generated (or, equivalently, the first iteration of the boolean fractal). Note that, in the row 0 we have 1 red/False square, in row 1 we have 3 of them and in row 2 we have 2 Green/True squares. This numbers are defined by the generator of the fractal, $P(z) = 1 + 3z - 2z^2$, and the colors (or boolean values), by the sign of the coefficients.

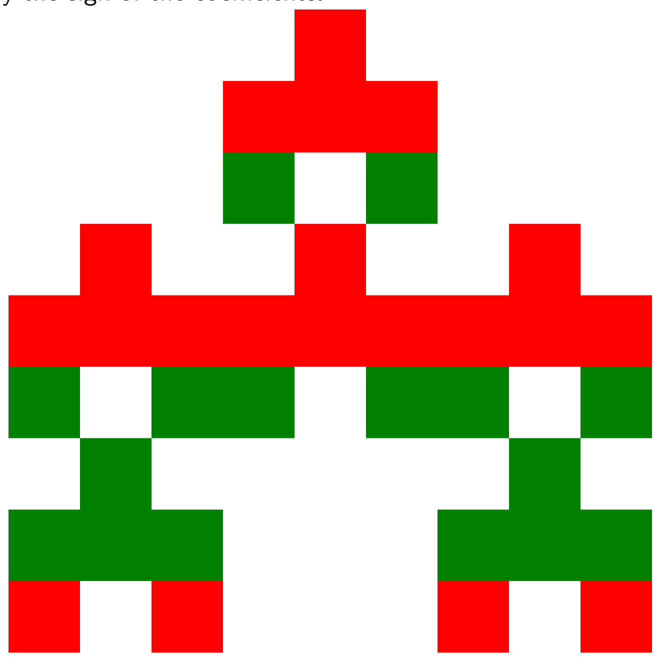

Figure 9: Second iteration of the Collatz boolean fractal. Note that it consists of 1 copy of the previous iteration, followed by 3 copies underneath, and followed by 2 more copies at the bottom with inverted colors, respecting the configuration of the pattern from where they originate. The number of copies, again, are defined by the generator $P(z) = 1 + 3z - 2z^2$, and the inversion (or not) of the colors, by the sign of the coefficients.

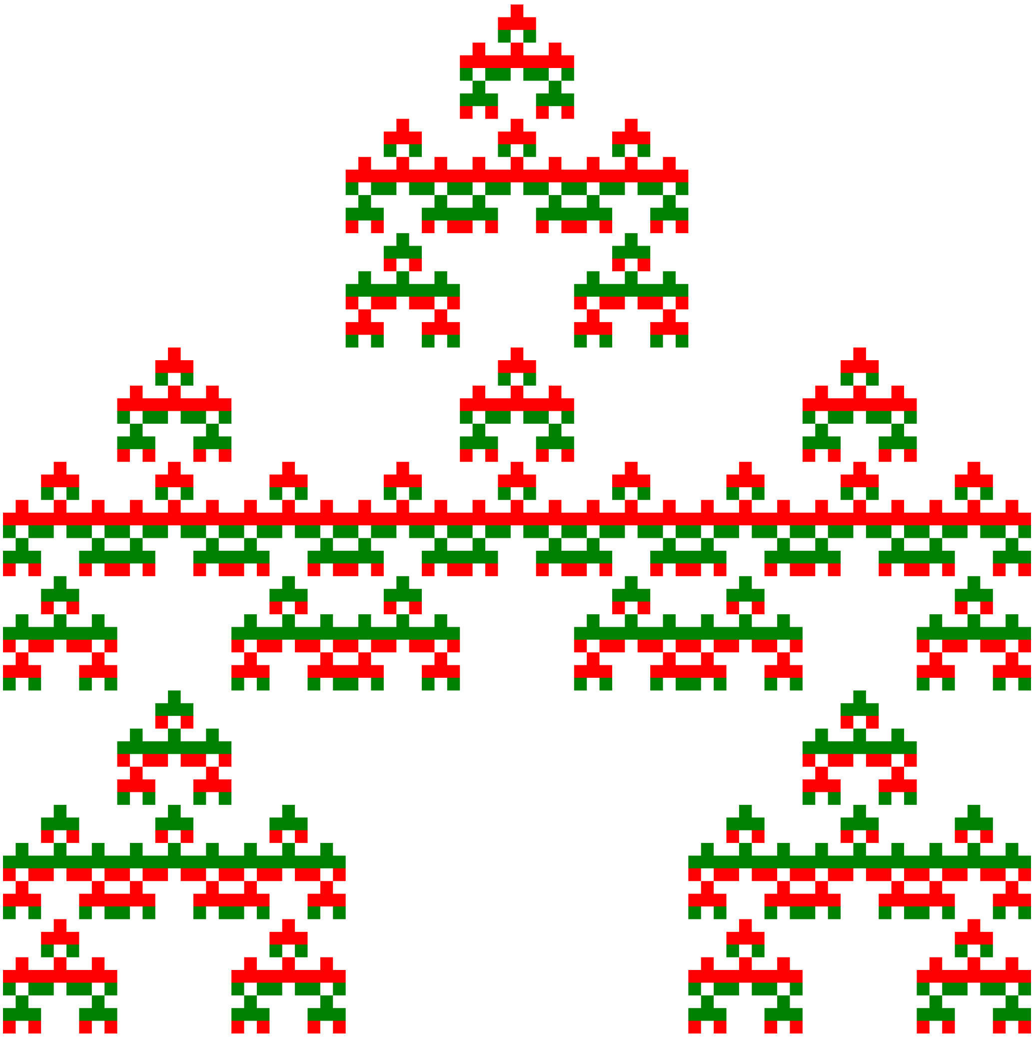

Figure 10: Fourth iteration of the Collatz boolean fractal

Observation 6.7. As we can see, the geometry of the main pattern can have different forms. For example, we can define an equivalent fractal as can be seen in Figure 11. The number of elements of each of the rows and their colour is the same as in the case of the Collatz boolean fractal.

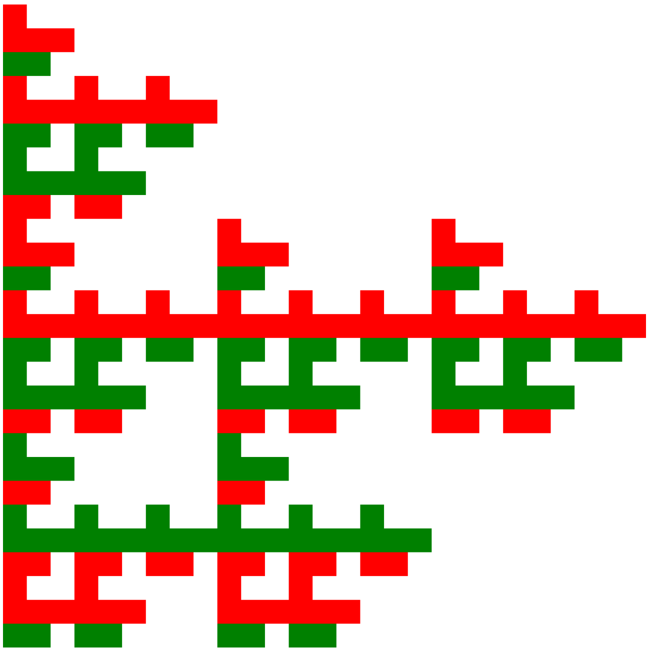

Figure 11: Third iteration of a fractal which is algebraically equivalent to Collatz boolean fractal, since it also comes generated by [1, 3, -2]

7. Some other approaches/Further work

In this section, we will discuss, without delving too deep, some other interesting approaches to tackle the Collatz conjecture which are worth discussing. The first of them, which is original in this work, regards the behaviour of the cycles of a family of functions which are similar to Collatz's, purely from experimental data. After this, we will discuss how some functional equations can define if the Collatz conjecture is true or false by giving the most straightforward example, and we will end discussing a little bit how transcendence theory could be used to prove or disprove the Collatz conjecture.

7.1 Cycle behaviour

One of the most common questions that may arise when studying the Collatz conjecture is about how would the paradigm change if the function with which we perform the iterations is slightly modified. So, let us try to see what happens when we change the $\frac{3n+1}{2}$ step to a $\frac{3n+a}{2}$ step for some odd integer a.

Definition 7.1. Let a be an odd integer. We will say $C_a(n) : \mathbb{Z} \to \mathbb{Z}$ is the function defined by:

$$C_a(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+a}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Observation 7.2. Let us make some obvious remarks with respect to this new definition. In the first place, it is clear that $C_1(n) = C(n)$. To continue, it is clear that the fact that a is odd forces that the image of any integer is an integer. We will have to take in account that the image of $C_a(n)$, for $n > 0$, will not necessarily be a positive integer in the case $a < 0$, although it will always be positive if $a > 0$.

Definition 7.3. Let n be an integer and a be an odd number. Denoting again by $C_a^k(n)$ the k-fold composition of the function $C_a(n)$, we will call the sequence $(n, C_a(1), C_a^2(n), C_a^3(n), \ldots)$ the pseudocollatz sequence of n under C_a.

We will try to focus now on the cycle structure of these pseudocollatz sequences. A logic thing to do would be to use the minimum element of the cycle as its representative. Therefore, we may come up with the following definition:

Definition 7.4. Let a be an odd integer and let n be a positive integer. We will say that the pair (a, n) is a Collatz attractor if there exists $k \in \mathbb{Z}_{>0}$ such that $C_a^k(n) = n$, and such that n is the minimum value in its pseudocollatz sequence under C_a, namely, $\min(n, C_a(1), C_a^2(n), C_a^3(n), \ldots) = n$.

Observation 7.5. As $C_a^k(n) = n$, $\min(n, C_a(1), C_a^2(n), C_a^3(n), \ldots) = \min(n, C_a(1), C_a^2(n), \ldots C_a^{k-1}(n))$.

Now we may ask ourselves the following question: how does the plot of the Collatz attractors look like? We may see it in Figure 12.

As we may see in the graph, the set of attractors appears to distribute its elements into a set of lines of the form $a_i(n) = p_i n + q_i$, with $a_i : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, and with $p_i, q_i \in \mathbb{Q}$. This set of lines appears to have a chaotic behaviour, but with some patterns emerging from this chaos. For example, we see that elements of the form $(a, 0)$ and $(a, -a)$, (a, a) and $(a, -5a)$ are of course Collatz Attractors since, for an odd value of a, $C_a(0) = 0$, $C_a(-a) = \frac{3(-a)+a}{2} = -a$, and $C_a^2(a) = C(C_a(a)) = C(\frac{3a+a}{2}) = C(2a) = a$, and $C_a^3(-5a) = C^2(\frac{3(-5a)+a}{2}) = C^2(-7a) = C(\frac{3(-7a)+a}{2}) = C(-10a) = -5a$, respectively.

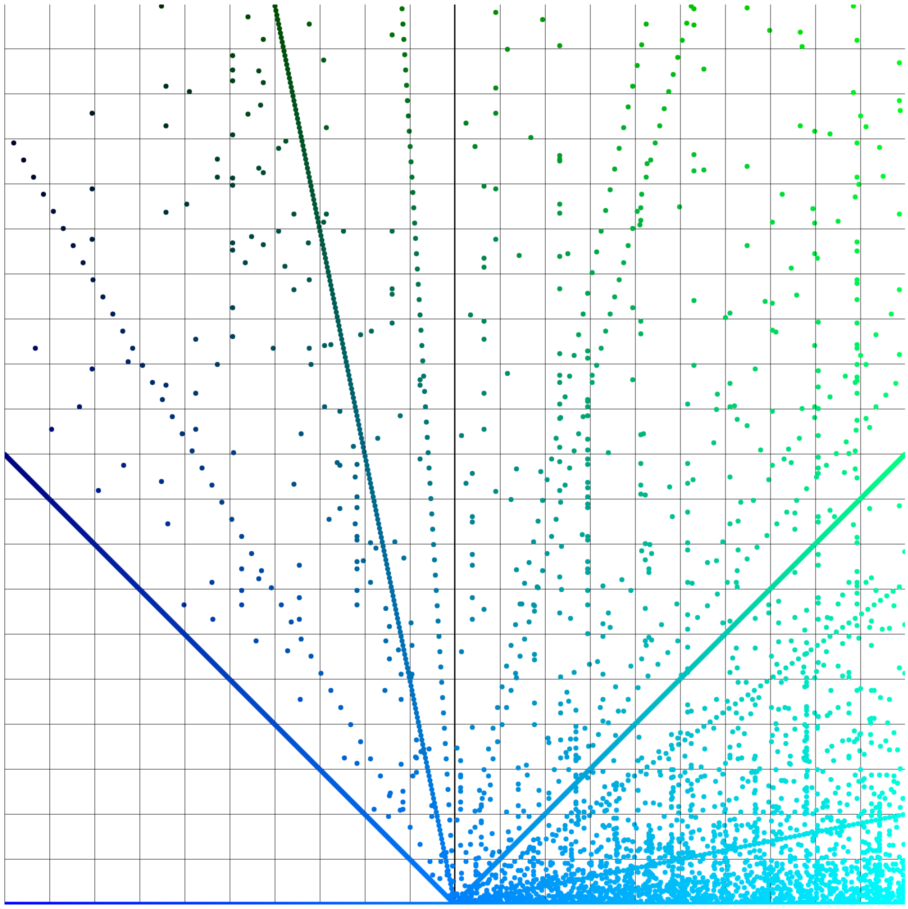

Figure 12: Plot of the Collatz Attractors (a, n), where a is represented in the horizontal axis and n is represented in the vertical axis. Here, the horizontal and vertical bars are distanced 100 units apart from each other. The bottom part of the graph represents $n = 0$ whereas the bold vertical bar in the middle leaves the positive values of a on the right and the negative ones on the left

Although we will not delve deep into this approach since it is out of the main scope of this work, it is probably worth investigating its properties, specially those regarding the distribution of slopes and the density and regularity of the lines, and how the distribution of the plot would change if instead of taking the minimum value of the cycle we would take the maximum, or all of them.

Finally, the fact that the lines seem to converge to either $(-1, 1)$ or $(1, 1)$ (and the fact that they look to converge to the first point if $a < 0$ and to the second if $a > 0$), and taking in account that the Collatz attractor $(1, 1)$ corresponds to the trivial cycle of the iterated Collatz function, this may mean that the Collatz function is special in some way among the family of functions $C_a(n)$.

7.2 Functional equations on generating functions

One very interesting way to tackle the conjecture is by using generating functions, that will result in some interesting functional equations. The nature of the family of solutions of such functional equations encode equivalent ways to formulate the Collatz problem.

Let us remember the Collatz tree we have seen at the introduction (Figure 1). As we said earlier, the Collatz conjecture is true if and only if the graph is weakly connected. If there is a counterexample for the conjecture, namely a non-trivial cycle or a divergent path, this will result, in the graph, in a disjoint weakly connected component that will include all the numbers that fall into such cycle or divergent path. In this case, it will be also convenient to consider the trivial cycle (0). Therefore, we can think of the following. Let us label each node of the graph n, namely, each non-negative integer, with some complex number f_n. Then, we will give the following rigidity conditions: $f_{2n} = f_n, f_{2n+1} = f_{3n+2}$. In other words, if one integer is the image (or an antiimage) of the other under the Collatz function, they will have the same label. Therefore, each number of a weakly connected component will have the same label.

Let us define now the generating function $F(z) = \sum_{k=0}^{\infty} f_n z^n$. But, before we continue, we may find useful the following proposition and corollary:

Proposition 7.6. *Let $G(z) = \sum_{k=0}^{\infty} g_k z^k \in \mathbb{C}[[z]]$. Then, for a given positive n, the generating function $G_{(n)}(z) = \sum_{k=0}^{\infty} g_{kn} z^{kn}$ satisfies:*

$$G_{(n)}(z) = \frac{1}{n} \sum_{r=0}^{n-1} G(\zeta_n^r z)$$

Where $\zeta_n = e^{\frac{2\pi i}{n}}$ is the n-th root of unity

Proof. We can see that

$$[z^m] G_{(n)}(z) = \frac{1}{n} \sum_{r=0}^{n-1} g_m (\zeta_n^r)^m = \frac{1}{n} g_m \sum_{r=0}^{n-1} (\zeta_n^m)^r.$$

Now, let $f(z) = z^{n-1} + z^{n-2} + \dots + 1$. As its roots are $\zeta_n^1, \dots, \zeta_n^{n-1}$, and $f(1) = n$, we will have:

$$\sum_{r=0}^{n-1} (\zeta_n^m)^r = f(\zeta_n^m) = \begin{cases} 0 & \text{if } n \nmid m \\ n & \text{if } n \mid m \end{cases}$$

leading to

$$[z^m] G_{(n)}(z) = \begin{cases} 0 & \text{if } n \nmid m \\ g_m & \text{if } n \mid m \end{cases}$$

and proving the equality. □

Corollary 7.7. *Let $G(z) = \sum_{k=0}^{\infty} g_k z^k \in \mathbb{C}[[z]]$. Then, for a given positive n, and an integer $d \in \{0, \dots, n-1\}$, the generating function $G_{d|n}(z) = \sum_{k=0}^{\infty} g_{kn+d} z^{kn+d}$ satisfies:*

$$G_{d|n}(z) = \frac{1}{n} \sum_{r=0}^{n-1} \zeta_n^{-dr} G(\zeta_n^r z)$$

Proof. Let $H(z) = z^{n-d} G(z)$. Clearly $H(z) = \sum_{k=0}^{\infty} h_k z^k$, where $h_k = 0$ if $k < n - d$ and $h_k = g_{k-(n-d)}$ otherwise. So, on the one hand:

$$H_{(n)}(z) = \sum_{k=0}^{\infty} h_{nk} z^{nk} = \sum_{k=1}^{\infty} h_{nk} z^{nk} = \sum_{k=1}^{\infty} g_{nk-n+d} z^{nk} =$$

$$\sum_{m=0}^{\infty} g_{nm+d} z^{n(m+1)} = z^{n-d} \sum_{m=0}^{\infty} g_{nm+d} z^{nm+d} = z^{n-d} G_{d|n}(z)$$

And on the other, as we have seen in the previous proposition:

$$H_{(n)}(z) = \frac{1}{n}\sum_{r=0}^{n-1} H(\zeta_n^r z) = \frac{1}{n}\sum_{r=0}^{n-1}(\zeta_n^r z)^{n-d} G(\zeta_n^r z) = z^{n-d}\frac{1}{n}\sum_{r=0}^{n-1}(\zeta_n^{n-d})^r G(\zeta_n^r z) = z^{n-d}\frac{1}{n}\sum_{r=0}^{n-1}\zeta_n^{-dr} G(\zeta_n^r z)$$

completing the proof. □

Now, bearing in mind that $f_{2n} = f_n, f_{2n+1} = f_{3n+2}$, we have the following. On the one hand:

$$F_{0|2}(z) = F_{(2)}(z) = \sum_{k=0}^{\infty} f_{2n} z^2 n = \sum_{k=0}^{\infty} f_n z^2 n = F(z^2)$$

On the other:

$$F_{1|2}(z) = \sum_{k=0}^{\infty} f_{2n+1} z^{2n+1} = \sum_{k=0}^{\infty} f_{3n+2} z^{2n+1} = z^{-\frac{1}{3}} \sum_{k=0}^{\infty} f_{3n+2} z^{\frac{2}{3}(3n+2)} = z^{-\frac{1}{3}} F_{2|3}(z^{\frac{2}{3}}) = \frac{1}{3z^{\frac{1}{3}}} \sum_{r=0}^{2} \zeta_3^r F(\zeta_3^r z^{\frac{2}{3}})$$

Finally, as $F_{0|2}(z) + F_{1|2}(z) = F(z)$, changing z to z^3 in order to get rid of the fractionary exponents, and letting $\omega = \zeta_3 = \frac{-1+\sqrt{3}i}{2}$, we get to the following result:

$$F(z^3) = F(z^6) + \frac{1}{3z}\sum_{r=0}^{2}\omega^r F(\omega^r z^2)$$

Now, if and only if the Collatz is true, we will have that $f_1 = f_2 = f_3 = ...$ As f_0 does not have any restriction, $F(z)$ will only be able to be of the form $F(z) = f_0 + \frac{f_1}{1-z} = \frac{a+bz}{1-z}$, for some complex numbers a, b such that $f_0 + f_1 = a, -f_0 = b$. It is easy to check that such solutions satisfy the previous functional equation. So, we can come up with the following equivalence of the Collatz conjecture:

Proposition 7.8. *The Collatz conjecture is true if and only if the set $\mathcal{F} \subset C[[z]]$ of solutions of the functional equation*

$$F(z^3) = F(z^6) + \frac{1}{3z}\sum_{r=0}^{2}\omega^r F(\omega^r z^2)$$

is the family of functions $\left\{\frac{a+bz}{1-z} : a, b \in \mathbb{C}\right\}$.

This approach we just covered was first introduced by Berg and Meinardus [12], [13].

7.3 Transcendence Theory

Transcendence theory (or transcendental number theory) is branch of mathematics study the properties of irrational numbers, and more in particular, those of transcendental numbers, which are those that are not a solution of a polynomial with rational coefficients.

The foundation of the theory begins with the study of diophantine approximation, which tries to approximate real numbers by rational numbers. The first well-known result in this field is the following:

Theorem 7.9 (Dirichlet's theorem on Diophantine Approximation)**.** *Let α be an irrational number. Then, there exist infinitely many rationals $\frac{p}{q}$ for which*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Remark 7.10. This theorem can be improved in the following sense. Émile Borel showed that, for each $c \geq \frac{1}{\sqrt{5}}$, one has that

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$$

is still satisfied for infinitely many rationals $\frac{p}{q}$. However, this is not true for smaller values of c, and a counterexample for that is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

Another very important result in this field is Roth's theorem:

Theorem 7.11 (Roth)**.** *Let α be a real, algebraic and irrational number. Then, $\forall \epsilon > 0$ there exist only finitely many rationals $\frac{p}{q}$ for which*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

or, equivalently, there is an ineffective (which means that no proof gives a method of computing it) positive constant $c(\alpha, \epsilon)$ for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{c(\alpha, \epsilon)}{q^{2+\epsilon}}$$

for any rational $\frac{p}{q}$.

Transcendence theory also gives us these two following important theorems, which are central in the theory. As an interesting fact, the first one is the (affirmative) answer to Hilbert's seventh problem:

Theorem 7.12 (Gelfond–Schneider)**.** *Let $a, b \in \bar{\mathbb{Q}}$ such that $a \notin \{0, 1\}, b \notin \mathbb{Q}$, then a^b (for any of its possible choices, since it is multivalued) is transcendental.*

Theorem 7.13 (Lindemann-Weierstrass)**.** *If $\alpha_1, \dots, \alpha_n$ are pairwise different algebraic numbers, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are linearly independent over $\bar{\mathbb{Q}}$ (the set of algebraic numbers), or equivalently, The set of solutions of $\beta_1 e^{\alpha_1} + \dots + \beta_n e^{\alpha_n} = 0$, with $\beta_1, \dots, \beta_n \in \bar{\mathbb{Q}}$, only has the trivial solution $\beta_1 = \dots = \beta_n = 0$.*

One of the most interesting consequences of Lindemann-Weierstrass theorem is the following corollary:

Corollary 7.14. *e and π are transcendental.*

Proof. If e was algebraic, e would be a solution of some polynomial with rational coefficients. Assume it exists and it has degree n, and rational coefficients $\beta_0 \dots, \beta_n$, with $\beta_n \neq 0$. Then, $\sum_{k=0}^{n} \beta_k e^k = 0$, which contradicts Lindemann-Weierstrass theorem.

If π was algebraic, then the field $\mathbb{Q}(\pi, i)$ would be inside the field of algebraic numbers $\bar{\mathbb{Q}}$, and it would contain the complex number πi. By Lindemann-Weierstrass, letting $\alpha_1 = 0, \alpha_2 = \pi i$, the equation $0 = \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} = \beta_1 - \beta_2$, has some non-zero solution, contradicting that πi is algebraic, and thus contradicting π is algebraic. $\square$

Once we have had this first glimpse into transcendence theory, we can present one of its most important results, Baker's theorem, which was stated and developed by Alan Baker in [19], [20], [21].

Theorem 7.15 (Baker)**.** *Let* $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{Q}} \setminus \{0\}$ *such that* $\log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$*. Then* $1, \log \alpha_1, \dots, \log \alpha_n$ *is linearly independent over* $\bar{\mathbb{Q}}$*.*

It is not difficult to show, although we will not do that here, that Baker's theorem implies the following generalization of the Gelfond–Schneider's theorem:

Corollary 7.16 (Generalization of Gelfond–Schneider)**.** *Let* $a_1, \dots, a_n, b_1, \dots, b_n$ *be algebraic complex numbers such that* $a_i \notin \{0, 1\}, \forall i$*, and such that,* $1, b_1, \dots, b_n$ *are linearly independent over* $\mathbb{Q}$*. Then* $a_1^{b_1} \cdots a_n^{b_n}$ *(for any of its possible choices, since it is multivalued) is transcendental.*

The reformulation of the weak Collatz conjecture made by Böhm and Sontacchi (Conjecture 4.2), talks about one kind of numbers having a divisor of the form $2^b - 3^{k-1}$. Therefore, it might be very important to study the properties of separation of powers of 2 and powers of 3.

First of all, we can see the following consequence of the Gelfond-Schneider theorem:

Corollary 7.17. $\frac{\log 2}{\log 3} \notin \bar{\mathbb{Q}}$

Proof. If $\frac{\log 2}{\log 3}$ was rational, it would be equal to $\frac{p}{q}$, with $p > 0, q > 0, p \neq q$, since $\frac{\log 2}{\log 3}$ is positive and different from 1. This would mean that $3^p = 2^q$, which is not possible since the LHS is odd and the RHS in even.

Now, we can apply Gelfond-Scheider. If $b = \frac{\log 2}{\log 3}$ was algebraic, setting $a = 3$, we would get $a^b = 3^{\frac{\log 2}{\log 3}} = 2$, which is not transcendental, contradicting the theorem. □

In [5], we can see a particular case of Baker's theorem that might be specially useful for the Collatz conjecture:

Proposition 7.18. *For any integers* p, q*, with* q *positive,*

$$\left| \frac{\log 2}{\log 3} - \frac{p}{q} \right| \geq \frac{c'}{q^c}$$

for some effective (namely, there exists a method to compute them) positive constants c, c' *which do not depend on* p *or* q*.*

By seeing that $3^p - 2^q = 3^p \left(1 - 3^{q\left(\frac{\log 2}{\log 3} - \frac{p}{q}\right)}\right)$ and doing some approximations, we can come up with the following corollary:

Corollary 7.19. *For any positive integers* p, q*,*

$$\left| \frac{\log 2}{\log 3} - \frac{p}{q} \right| \geq \frac{C'}{q^C} 3^p$$

for some effective positive constants C, C'*, different from those of the previous proposition, which do not depend on* p *or* q*.*

Taking in account that $|3^p - 2^q| \geq 1$, we can achieve the following bound by plugging this inequality into the identity $3^p - 2^q = 3^p \left(1 - 3^{q\left(\frac{\log 2}{\log 3} - \frac{p}{q}\right)}\right)$ we have seen before:

Proposition 7.20. *For any integers p, q, with $q > 0$*

$$\left|\frac{\log 2}{\log 3} - \frac{p}{q}\right| \geq \frac{k}{2^q}$$

for some effective positive constant k, which do not depend on p or q.

Finally, one last interesting strong result that can be deduced from Baker's theorem would be the following:

Proposition 7.21. *For any integers p, q, with $q > 1$*

$$\left|\frac{\log 2}{\log 3} - \frac{p}{q}\right| \geq e^{-K' \log^K q}$$

for some effective positive constants K, K', which do not depend on p or q.

8. Conclusions

In this work, we have introduced a new equivalent formulation for the weak Collatz conjecture (which states that no cycle in a Collatz sequence, which is the sequence of numbers obtained by iterating the Collatz function starting from some initial positive integer, is non-trivial). To do that, we have introduced the concept of algebraic fractals.

We have defined the concept of algebraic fractals, and from there, the one of self-similar algebraic fractals. In the latter case, we have seen some properties about their coefficients, like what structure do they have and their relation to tensors. In general, we have also seen the analytic properties of algebraic fractals, and some properties of existence and uniqueness. We have also defined the concept of fractal sum and have proved the recurrence formula that their coefficients follow.

From there, we have also defined what integer fractals are, which are algebraic fractals the coefficients of which are all non-negative integers, and what boolean fractals are, which are the same but dropping the condition of non-negativity. We have also given a geometric interpretation for both of them.

Finally, by exploiting the similarities of a counting problem in the Sierpinski's triangle with a reformulation of the weak Collatz conjecture due to Böhm and Sontacchi, and with the aid of the theory of algebraic fractals developed previously, we have been able to give a reformulation of the weak Collatz conjecture, both in terms of the algebraic properties of self-similar boolean algebraic fractals and in terms of their geometric ones.

The main contributions given here, in conclusion, are on the one hand, the concept of algebraic fractals, alongside with its particular cases, which probably can have some interesting applications in Mathematics, specially in the field of Combinatorics, and on the other, a reformulation of the weak Collatz conjecture based on an object which is very simple to construct iteratively, which has both a geometrical and an algebraic meaning, and that does not have a very chaotic behaviour. In other words, we have reduced the problem of finding non-trivial cycles in Collatz sequences to a problem about counting in a geometric object the construction of which is extremely easy to understand.

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A. Code

In this appendix we will include some code that either may be at some extent interesting to study the Collatz conjecture (which will be the code made in C++), or either has been used to generate some of the images used in the work (which are the ones made with Python3 and the PIL Library, which allows to easily create images made up with elementary shapes). All the code has been made by the author of this work.

A.1 Python3

Attractors image This program generates Figure 12.

```
from PIL import Image, ImageDraw
import math

MAX = 999;        # a will iterate from -MAX to MAX
MAX2 = 1999;      # n will iterate from 0 to MAX2 - 1
MAXITER = 10000;

img = Image.new('RGB', [2*MAX + 1, MAX2], 'White')
dib = ImageDraw.Draw(img)

def Col(n, a) :
    return n/2 if n%2 == 0 else (3*n + a)/2;

# Tests if (a, n) is an attractor
def test(a, n) :
    it = 0;
    x = n;
    while it < MAXITER and x >= n :
        it = it + 1;
        x = Col(x, a);
        if x == n :
            return True;

    return False;

# Draws rectangles (used to draw vertical/horizontal lines)
def draw_rectangle (x1, y1, x2, y2, col) :
    dib.polygon([(x1, y1), (x1, y2), (x2, y2), (x2, y1)], col)

draw_rectangle(MAX - 1, 0, MAX + 1, MAX2 - 1, 'Black')

cnt = 0;
while (cnt + 100 < MAX2) :
    cnt = cnt + 100
    draw_rectangle(0, MAX2 - 1 - cnt, 2*MAX, MAX2 - 1 - cnt, 'Black')


for a in range(-MAX, MAX+1, 2):
    print(a)
    attractors = []
    if (MAX+a)%100 == 0 :
        draw_rectangle(MAX + a, 0, MAX + a, MAX2 - 1, 'Black')
```

```
    for n in range(MAX2) :
        if test(a, n):
            attractors.append(n);
    for x in attractors :
        X = a+MAX
        Y = MAX2-1-x
        r = 0
        g = 255*X//(2*MAX + 1)
        b = 255*Y//MAX2
        dib.ellipse([X-5, Y-5, X+5, Y+5], (r, g, b)) # Draws point
        #dib.point((X, Y), (r, g, b))


img.save("attractors.png")
img.show()
```

Fractal Generator (using squares) This program generates a fractal with a given pattern. Its input starts with k, n, m, where k is the number of iterations and $n \times m$ is the size of the initial pattern. Then it asks for a $n \times m$ matrix, made up of F's, T's and dots, for False elements, True elements, and empty elements, respectively. It has been used to generate Figures 5, 7, 8, 9, 10 and 11. The following input generates Figure 10.

```
4
3
3
.F.
FFF
T.T
```

```
import math
from PIL import Image, ImageDraw

def rect (x1, y1, x2, y2, col): # Function to draw a rectangle
  dib.polygon([(x1, y1), (x2, y1), (x2, y2), (x1, y2)], col)

def triangle (x1, y1, x2, y2, col):
        dib.polygon([(x1, y1), (x2, y2), (x1, y2)], col)

k = int(input()) # Number of iterations
n = int(input()) # Size (horizontal) of main pattern
m = int(input()) # Size (vertical) of main pattern

M = []

for i in range (m):
  M.append(input())

#Size of the squares
sx = 1
sy = 1
SIZELOWERBOUNDX = 2000
SIZELOWERBOUNDY = 2000
while sx*(n**k) < SIZELOWERBOUNDX:
     sx = sx*n
```

```
while sy*(m**k) < SIZELOWERBOUNDY:
    sy = sy*n


img = Image.new('RGB', (sx*(n**k), sy*(m**k)), 'White')
dib = ImageDraw.Draw(img)

def fractal (x, y, k, b): # Recursive function to generate the fractal
  if k == 0:
    col = 'Red'
    if b:
      col = 'Green'
    rect(sx*x,sy*y,sx*x+sx-1,sy*y+sy-1, col)
    return
  for j in range (m):
    for i in range (n):
      if M[j][i] == 'F':
        fractal(x + i*(n**(k-1)), y + j*(m**(k-1)), k-1, b)
      if M[j][i] == 'T':
        c = True
        if b:
          c = False
        fractal(x + i*(n**(k-1)), y + j*(m**(k-1)), k-1, c)

fractal(0, 0, k, False)

img.save('output.png')
img.show()
```

Fractal Generator (using triangles) Same as before, but now it uses triangles. It has been used to generate Figures 4 and 6. The following input generates Figure 6.

```
4
2
2
F.
FF
```

```
import math
from PIL import Image, ImageDraw

def rect (x1, y1, x2, y2, col):
  dib.polygon([(x1, y1), (x2, y1), (x2, y2), (x1, y2)], col)

def triangle (x1, y1, x2, y2, col):
        dib.polygon([(x1, y1), (x2, y2), (x1, y2)], col)

def gcd (a, b):
  if b == 0:
    return a
  else:
    return gcd (b, a%b)

dx = [-1, 0, 1, 0]
dy = [0, -1, 0, 1]
```

```python
#c1 = input()
#c2 = input()
#c1 = 'White'
#c2 = 'Black'

k = int(input())
n = int(input())
m = int(input())

M = []

for i in range (m):
  M.append(input())

#Size of the triangles
sx = 1
sy = 1
SIZELOWERBOUNDX = 2000
SIZELOWERBOUNDY = 2000
while sx*(n**k) < SIZELOWERBOUNDX:
    sx = sx*n

while sy*(m**k) < SIZELOWERBOUNDY:
    sy = sy*n


img = Image.new('RGB', (sx*(n**k), sy*(m**k)), 'White')
dib = ImageDraw.Draw(img)

def fractal (x, y, k, b):
  if k == 0:
    col = 'Red'
    if b:
      col = 'Green'
    #dib.point((x,y), col)
    triangle(sx*x,sy*y,sx*x+sx-1,sy*y+sy-1, col)
    return
  for j in range (m):
    for i in range (n):
      if M[j][i] == 'F':
        fractal(x + i*(n**(k-1)), y + j*(m**(k-1)), k-1, b)
      if M[j][i] == 'T':
        c = True
        if b:
          c = False
        fractal(x + i*(n**(k-1)), y + j*(m**(k-1)), k-1, c)


fractal(0, 0, k, False)

img.save('output.png')
img.show()
```

A.2 C++

Attractors. This program does the following. Using the parameter MAX, and odd positive integer, and $MAX2$ a positive integer, finds all Collatz attractors (a, n), with $|a| \leq MAX$ and $0 \leq n \leq MAX2$. Then, it writes in two comma-separated value files. In `collatz_cycles.csv`, it writes many lines, one for each odd value of a in the mentioned range, starting by the number a, followed by all the values n such that (a, n) is a Collatz attractor. In `ratios.csv`, it writes all the found reduced fractions of $\frac{n}{a}$.

```cpp
#include <bits/stdc++.h>
using namespace std;

typedef long long int LL;
typedef vector<LL> VI;
typedef vector<VI> VVI;
typedef pair<LL, LL> PII;

const int MAX = 999;
const int MAX2 = 1000000;
const int MAXITER = 10000;

LL Col(LL n, LL a) {
    return n%2 == 0 ? n/2 : (3*n + a)/2;
}

// Tests if (a, n) is a Collatz attractor
bool test(LL a, LL n) {
    int it = 0;
    LL x = n;
    while (it++ < MAXITER and x >= n) {
        x = Col(x, a);
        if (x == n) return true;
    }
    return false;
}

LL gcd(LL a, LL b) {
    return b ? gcd(b, a%b) : a;
}

// Returns the reduced form of a/b, with b > 0
PII normalize(LL a, LL b) {
    if (b < 0) a = -a, b = -b;
    bool neg = false;
    if (a < 0) neg = true, a = -a;
    LL g = gcd(a, b);
    a/=g, b/=g;
    if (neg) a = -a;
    return {a, b};
}

struct comp {
  bool operator() (const PII& lhs, const PII& rhs) const
  {return lhs.first*rhs.second<lhs.second*rhs.first;}
};
map<PII, int, comp> M;
```

```
int main() {
    assert(MAX%2 and MAX > 0);
    ofstream of("collatz_cycles.csv");
    VVI attractors(2*MAX + 1);;
    for (LL a = -MAX; a <= MAX; a+=2) {
        cerr << a << endl;
        for (LL n = 0; n <= MAX2; ++n) {
            if (test(a, n)) {
                attractors[MAX + a].push_back(n);
                if (a) ++M[normalize(n, a)];
            }
        }
    }
    for (LL a = -MAX; a <= MAX; ++a) {
        of << a;
        for (LL x : attractors[MAX + a]) of << ',' << x;
        of << endl;
    }
    of.close();
    of = ofstream("ratios.csv");
    for (auto x : M) {
        of << x.first.first << '/' << x.first.second << " = " <<
        double(x.first.first)/x.first.second << '\t' << x.second << endl;
    }
    of.close();
}
```

Number of elements of the n-th row of the Collatz boolean fractal, with n having no 2's in it base 3 expression. Given a parameter MAX, this program gives the number in the row n of the Collatz boolean fractal, for each $n < MAX$ that have no 2's in its base 3 representation. One may note that the output is the sequence OEIS A119733.

```
#include <bits/stdc++.h>
using namespace std;

typedef long long int LL;

// Returns how many times the value v appears
// in the base b representation of x
LL f(LL v, LL x, LL b){
    return (x ? LL(x%b == v) + f(v, x/b, b) : 0);
}

// Returns min p^k such that p^k > x
LL min_power(LL x, LL p) {
    int r = 1;
    while (r*p <= x) r*=p;
    return r;
}

// If a^k <= x <= a^{k+1}, returns b^k
LL convert_power(LL x, LL a, LL b) {
    LL r = 1;
    while (x > 1) {
        r*=b;
        x/=a;
    }
```

```
    return r;
}

LL S(LL x) {
    if (x == 0) return 0;
    LL mp = min_power(x, 3);
    return convert_power(mp, 3, 2) + 3*S(x - mp);
}

int main() {
    LL acum = 0;
    LL MAX = 125;
    for (LL x = 0; x < MAX; ++x) {
        if (f(2, x+1, 3)) continue;
        cerr << x << '\t' << S(x+1) << endl;
    }
}
```

Number of elements of the n-th row of the Collatz boolean fractal Same as before, but dropping the restriction on the base 3 expression of n.

```
#include <bits/stdc++.h>
using namespace std;

typedef long long int LL;
typedef vector<LL> VLL;

// Returns how many times the value v appears
// in the base b representation of x
LL f(LL v, LL x, LL b){
    return (x ? LL(x%b == v) + f(v, x/b, b) : 0);
}

// Returns min p^k such that p^k > x
LL min_power(LL x, LL p) {
    int r = 1;
    while (r*p <= x) r*=p;
    return r;
}

// If a^k <= x <= a^{k+1}, returns b^k
LL convert_power(LL x, LL a, LL b) {
    LL r = 1;
    while (x > 1) {
        r*=b;
        x/=a;
    }
    return r;
}

LL S(LL x, LL m) {
    if (x == 0) return 0;
    LL mp = min_power(x, 3);
    return convert_power(mp, 3, 2) + m*S(x - mp, m);

}
```

```
int main() {
    int MAX = 81;
    VLL v(MAX+1);
    v[0] = 1;
    for (int p = 1; p <= MAX; p *= 3) {
        for (int i = 0; i < p; ++i) {
            v[i + p] = 3*v[i];
            v[i + 2*p] = -2*v[i];
        }
    }
    LL acum = 0;
    for (int i = 0; i < v.size() - 1; ++i) {
        acum += v[i];
        cout << i << '\t' << v[i+1] << '\t' << acum << endl;
    }
}
```

Collatz ending. This program tells, for a number n, if the iteration sequence of $C^2(n)$ converges to 1 or to 2.

```
#include <bits/stdc++.h>
using namespace std;
typedef long long int LL;

LL Col(LL x){
    return ((x&1 ? 3*x+1 : x) >> 1);
}

// Returns Col(x) composed n times
LL Col(LL x, int n){
    return n ? Col(Col(x), n-1) : x;
}

int main(){
    const int N = 8;
    const int MAX = (1 << N);
    int c[] = {0,0};
    for (LL i = 1; i < MAX; i += 1){
        LL x = i;
        while ((x = Col(x, 2)) > 2);
        cout << i << '\t' << bitset<N>(i) << '\t' << (x == 1 ? 'X' : '.') << endl;
        ++c[x-1];
    }
    cout << c[0] << ' ' << c[1] << endl;
}
```

First iterations This program gives, given an N, for each n such that $1 \leq n \leq N$, the set of lines $C^n(2^n + r)$, with $0 \leq r < 2^n - 1$, as we have seen in Section 3. It also gives a string of bits, which, from right to left (only the last n bits), the i-th last is a 0, if it takes a $\frac{x}{2}$ step at the i-th iteration or 1 otherwise. Its output, for $N = 6$ is given in the next appendix.

```
#include <bits/stdc++.h>
using namespace std;
typedef long long int LL;
typedef pair<int, int> PII;

LL Collatz(LL x){
```

```
    return x&1 ? (3*x+1)/2 : x/2;
}

PII IC(int r, int a, int b){ // Gives line equation
  if (r == 0){
    return make_pair(a, b);
  }
  return b&1 ? IC(r-1, 3*a, Collatz(b)) : IC(r-1, a, Collatz(b));
}

int main(){
  const int N = 6;
  for (int i = 1; i <= N; ++i){
    const int r = i;
    for (int j = 0; j < (1 << i); ++j){
      int x = j;
      int y = 0;
      int z = 0;
      for (int k = 0; k < i; ++k){
        y = y + (bool(x&1) << k);
        z = 2*z + (bool(x&1));
        x = Collatz(x);
      }
            PII P = IC(i, 1, j);
      cout << setw(3) << (1 << i) << "n + " << setw(2) << j <<
            "\t==>\t" << setw(4) << P.first << "n + " << setw(3)
            << P.second << "\t\t";
      cout << bitset<N>(y) << '\t' << y << endl;
    }
    cout << endl << endl << endl;
  }
}
```

B. First iterations

In Theorem 3.1 we have seen the aspect of the Collatz function iterated many times. In the next table, each line has an equation of a line $2^k n + r$ and its image under C^k, that is, $C^k(2^k n + r)$. It follows a sequence of zeroes (which represent a $\frac{x}{2}$ step), and ones (which represent a $\frac{3x+1}{2}$ step). The rightmost bit represents the first iteration, the second from the right represents the second iteration, and the k-th from the right represents the k-th iteration. Those more to the left must be ignored. It follows a decimal representation of the number.

2n + 0	==>	1n + 0	000000	0
2n + 1	==>	3n + 2	000001	1

4n + 0	==>	1n + 0	000000	0
4n + 1	==>	3n + 1	000001	1
4n + 2	==>	3n + 2	000010	2
4n + 3	==>	9n + 8	000011	3

8n + 0	==>	1n + 0	000000	0
8n + 1	==>	9n + 2	000101	5
8n + 2	==>	3n + 1	000010	2
8n + 3	==>	9n + 4	000011	3
8n + 4	==>	3n + 2	000100	4
8n + 5	==>	3n + 2	000001	1
8n + 6	==>	9n + 8	000110	6
8n + 7	==>	27n + 26	000111	7

16n + 0	==>	1n + 0	000000	0
16n + 1	==>	9n + 1	000101	5
16n + 2	==>	9n + 2	001010	10
16n + 3	==>	9n + 2	000011	3
16n + 4	==>	3n + 1	000100	4
16n + 5	==>	3n + 1	000001	1
16n + 6	==>	9n + 4	000110	6
16n + 7	==>	27n + 13	000111	7
16n + 8	==>	3n + 2	001000	8
16n + 9	==>	27n + 17	001101	13
16n + 10	==>	3n + 2	000010	2
16n + 11	==>	27n + 20	001011	11
16n + 12	==>	9n + 8	001100	12
16n + 13	==>	9n + 8	001001	9

16n + 14	⟹	27n + 26	001110	14
16n + 15	⟹	81n + 80	001111	15

32n + 0	⟹	1n + 0	000000	0
32n + 1	⟹	27n + 2	010101	21
32n + 2	⟹	9n + 1	001010	10
32n + 3	⟹	9n + 1	000011	3
32n + 4	⟹	9n + 2	010100	20
32n + 5	⟹	9n + 2	010001	17
32n + 6	⟹	9n + 2	000110	6
32n + 7	⟹	81n + 20	010111	23
32n + 8	⟹	3n + 1	001000	8
32n + 9	⟹	81n + 26	011101	29
32n + 10	⟹	3n + 1	000010	2
32n + 11	⟹	27n + 10	001011	11
32n + 12	⟹	9n + 4	001100	12
32n + 13	⟹	9n + 4	001001	9
32n + 14	⟹	27n + 13	001110	14
32n + 15	⟹	81n + 40	001111	15
32n + 16	⟹	3n + 2	010000	16
32n + 17	⟹	9n + 5	000101	5
32n + 18	⟹	27n + 17	011010	26
32n + 19	⟹	27n + 17	010011	19
32n + 20	⟹	3n + 2	000100	4
32n + 21	⟹	3n + 2	000001	1
32n + 22	⟹	27n + 20	010110	22
32n + 23	⟹	27n + 20	000111	7
32n + 24	⟹	9n + 8	011000	24
32n + 25	⟹	27n + 22	001101	13
32n + 26	⟹	9n + 8	010010	18
32n + 27	⟹	81n + 71	011011	27
32n + 28	⟹	27n + 26	011100	28
32n + 29	⟹	27n + 26	011001	25
32n + 30	⟹	81n + 80	011110	30
32n + 31	⟹	243n + 242	011111	31

64n + 0	⟹	1n + 0	000000	0
64n + 1	⟹	27n + 1	010101	21
64n + 2	⟹	27n + 2	101010	42
64n + 3	⟹	27n + 2	100011	35
64n + 4	⟹	9n + 1	010100	20
64n + 5	⟹	9n + 1	010001	17

64n + 6	==>	9n + 1	000110	6
64n + 7	==>	81n + 10	010111	23
64n + 8	==>	9n + 2	101000	40
64n + 9	==>	81n + 13	011101	29
64n + 10	==>	9n + 2	100010	34
64n + 11	==>	27n + 5	001011	11
64n + 12	==>	9n + 2	001100	12
64n + 13	==>	9n + 2	001001	9
64n + 14	==>	81n + 20	101110	46
64n + 15	==>	81n + 20	001111	15
64n + 16	==>	3n + 1	010000	16
64n + 17	==>	27n + 8	100101	37
64n + 18	==>	81n + 26	111010	58
64n + 19	==>	81n + 26	110011	51
64n + 20	==>	3n + 1	000100	4
64n + 21	==>	3n + 1	000001	1
64n + 22	==>	27n + 10	010110	22
64n + 23	==>	27n + 10	000111	7
64n + 24	==>	9n + 4	011000	24
64n + 25	==>	27n + 11	001101	13
64n + 26	==>	9n + 4	010010	18
64n + 27	==>	243n + 107	111011	59
64n + 28	==>	27n + 13	011100	28
64n + 29	==>	27n + 13	011001	25
64n + 30	==>	81n + 40	011110	30
64n + 31	==>	243n + 121	011111	31
64n + 32	==>	3n + 2	100000	32
64n + 33	==>	81n + 44	110101	53
64n + 34	==>	9n + 5	001010	10
64n + 35	==>	9n + 5	000011	3
64n + 36	==>	27n + 17	110100	52
64n + 37	==>	27n + 17	110001	49
64n + 38	==>	27n + 17	100110	38
64n + 39	==>	243n + 152	110111	55
64n + 40	==>	3n + 2	001000	8
64n + 41	==>	243n + 161	111101	61
64n + 42	==>	3n + 2	000010	2
64n + 43	==>	81n + 56	101011	43
64n + 44	==>	27n + 20	101100	44
64n + 45	==>	27n + 20	101001	41
64n + 46	==>	27n + 20	001110	14
64n + 47	==>	243n + 182	101111	47
64n + 48	==>	9n + 8	110000	48
64n + 49	==>	9n + 7	000101	5
64n + 50	==>	27n + 22	011010	26
64n + 51	==>	27n + 22	010011	19

64n + 52	$\Longrightarrow$	9n + 8	100100	36
64n + 53	$\Longrightarrow$	9n + 8	100001	33
64n + 54	$\Longrightarrow$	81n + 71	110110	54
64n + 55	$\Longrightarrow$	81n + 71	100111	39
64n + 56	$\Longrightarrow$	27n + 26	111000	56
64n + 57	$\Longrightarrow$	81n + 74	101101	45
64n + 58	$\Longrightarrow$	27n + 26	110010	50
64n + 59	$\Longrightarrow$	81n + 76	011011	27
64n + 60	$\Longrightarrow$	81n + 80	111100	60
64n + 61	$\Longrightarrow$	81n + 80	111001	57
64n + 62	$\Longrightarrow$	243n + 242	111110	62
64n + 63	$\Longrightarrow$	729n + 728	111111	63